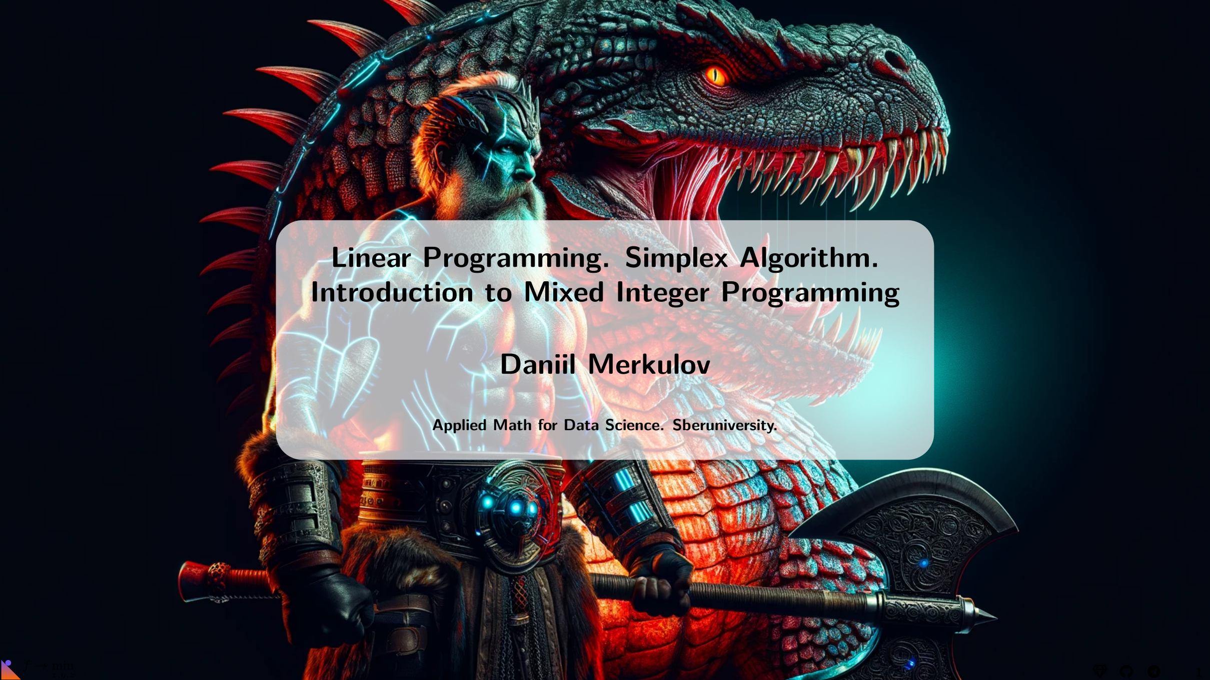

Linear Programming. Simplex Algorithm. Introduction to Mixed Integer Programming

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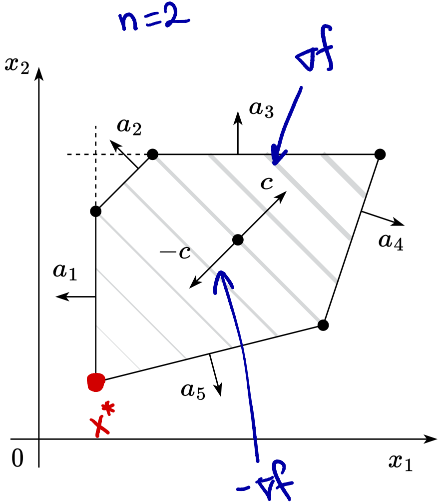
Applied Math for Data Science. Sberuniversity.

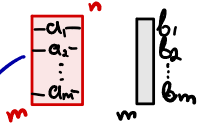
Linear Programming

What is Linear Programming?

мин. $f_0(x) = c^T x$ $\quad \nabla f(x) = c$ $\quad f: \mathbb{R}^n \to \mathbb{R}$



Generally speaking, all problems with linear objective and linear equalities/inequalities constraints could be considered as Linear Programming. However, there are some formulations.



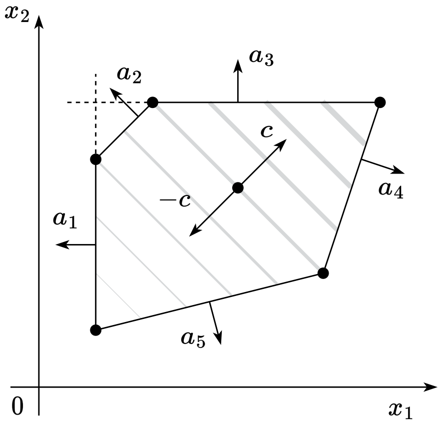
$$
\begin{split}
&\min_{x \in \mathbb{R}^n} c^\top x \\
\text{s.t. } & Ax \leq b
\end{split}
\tag{LP.Basic}
$$

for some vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and matrix $A \in \mathbb{R}^{m \times n}$. Where the inequalities are interpreted component-wise.

$a_i^T x \leq b_i \iff a_i^T x - b_i \leq 0 \iff f_i(x) = a_i^T x - b_i$

$\min f(x)$
$x \in \mathbb{R}^n$
$f_i(x) \leq 0$
$h_i(x) = 0$

What is Linear Programming?



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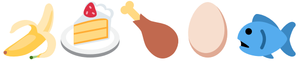
for some vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and matrix $A \in \mathbb{R}^{m \times n}$. Where the inequalities are interpreted component-wise.

Standard form. This form seems to be the most intuitive and geometric in terms of visualization. Let us have vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and matrix $A \in \mathbb{R}^{m \times n}$.

$$
\boxed{
\begin{aligned}
&\min_{x \in \mathbb{R}^n} c^\top x \\
\text{s.t. } & Ax = b \\
& x_i \geq 0, \; i = 1, \dots, n
\end{aligned}
}
\tag{LP.Standard}
$$

Example: Diet problem



p

Proteins

Carbs

Fats

Calories

Vitamin D

Amount per 100g

$W \in \mathbb{R}^{n \times p}$

W

n

$c \in \mathbb{R}^p$, price per 100g

$r \in \mathbb{R}^n$, nutrient requirements

$x \in \mathbb{R}^p$, amount of products, 100g

$$\boxed{\min_{x \in \mathbb{R}^p} c^T x}$$

$$\boxed{Wx \succeq r}$$

$$\boxed{x \succeq 0}$$

только покупаем

для каждого из p продуктов есть цена

$c \in \mathbb{R}^p$

x - кол-во купленных продуктов

$$\sum_{i=1}^{p} c_i \cdot x_i = c^T x$$

хочу, чтобы корзина $\geq$ 1500 ккал

$$\sum_{i=1}^{p} w_{cal,i} x_i \geq r_{cal}$$

$$(W \cdot x)_{cal} \geq r_{cal}$$

Example: Diet problem

$$
\begin{array}{c|c}
 & 🍌\ 🍰\ 🍗\ 🥚\ 🐟 \\
\begin{array}{c} \text{Proteins} \\ \\ \text{Carbs} \\ \\ \text{Fats} \\ \\ \text{Calories} \\ \\ \text{Vitamin D} \end{array} &
\left[\begin{array}{c} \\ \\ \\ \text{Amount per 100g} \\ \\ W \in \mathbb{R}^{n \times p} \\ \\ \\ \\ \end{array} \right]
\end{array}
$$

$c \in \mathbb{R}^p$, price per 100g

$r \in \mathbb{R}^n$, nutrient requirements

$x \in \mathbb{R}^p$, amount of products, 100g

$$
\min_{x \in \mathbb{R}^p} c^T x
$$

$$
Wx \succeq r
$$

$$
x \succeq 0
$$

Imagine, that you have to construct a diet plan from some set of products: bananas, cakes, chicken, eggs, fish. Each of the products has its vector of nutrients. Thus, all the food information could be processed through the matrix W. Let us also assume, that we have the vector of requirements for each of nutrients $r \in \mathbb{R}^n$. We need to find the cheapest configuration of the diet, which meets all the requirements:

$$
\boxed{
\begin{aligned}
&\min_{x \in \mathbb{R}^p} c^\top x \\
\text{s.t. } & Wx \succeq r \\
& x_i \geq 0, \ i = 1, \dots, n
\end{aligned}
}
$$

Open In Colab

$r_{min} \preceq Wx \preceq r_{MAX}$

Basic transformations

- Max-min

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^n} c^\top x \\
&\text{s.t. } Ax \leq b
\end{aligned}
\quad \leftrightarrow \quad
\begin{aligned}
&\max_{x \in \mathbb{R}^n} -c^\top x \\
&\text{s.t. } Ax \leq b
\end{aligned}
$$

Basic transformations

- Max-min

$$\begin{aligned}\min_{x \in \mathbb{R}^n} c^\top x \\ \text{s.t. } Ax \leq b\end{aligned} \quad \leftrightarrow \quad \begin{aligned}\max_{x \in \mathbb{R}^n} -c^\top x \\ \text{s.t. } Ax \leq b\end{aligned}$$

- Equality to inequality

$$Ax = b \leftrightarrow \begin{cases} Ax \leq b \\ Ax \geq b \end{cases}$$

$$h(x) = 0 \Leftrightarrow \begin{cases} h(x) \geq 0 \\ h(x) \leq 0 \end{cases}$$

$$-h(x) \leq 0$$

Basic transformations

- Max-min

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \; & c^\top x \\ \text{s.t. } & Ax \leq b \end{aligned} \quad \leftrightarrow \quad \begin{aligned} \max_{x \in \mathbb{R}^n} \; & -c^\top x \\ \text{s.t. } & Ax \leq b \end{aligned}$$

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$$Ax = b \leftrightarrow \begin{cases} Ax \leq b \\ Ax \geq b \end{cases}$$

- Inequality to equality by increasing the dimension of the problem by m.

$$Ax \leq b \leftrightarrow \begin{cases} Ax + z = b \\ z \geq 0 \end{cases}$$

$$Ax \leq b \Rightarrow \underbrace{b - Ax}_{z} \geq 0$$

$$z = b - Ax$$

$$Ax + z = b$$

Basic transformations

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- Unsigned variables to nonnegative variables.

x^+

$$
\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} x^-
$$

n $\quad$ $2n$

$$
x \leftrightarrow \begin{cases} x = x_+ - x_- \\ x_+ \geq 0 \\ x_- \geq 0 \end{cases}
$$

Example: Chebyshev approximation problem

$\|x\|_\infty = \max_i |x_i|$

$$
\boxed{\min_{x \in \mathbb{R}^n} \|Ax - b\|_\infty} \leftrightarrow \min_{x \in \mathbb{R}^n} \max_i |a_i^T x - b_i|
$$

Could be equivalently written as an LP with the replacement of the maximum coordinate of a vector:

Example: Chebyshev approximation problem

i-ая коорд. вектора Ax-b

t_1 2

$|a_i^T x - b_i| = t_i$ 5

t_m 7

$$
\min_{x \in \mathbb{R}^n} \|Ax - b\|_\infty \leftrightarrow \min_{x \in \mathbb{R}^n} \max_i |a_i^T x - b_i|
$$

Could be equivalently written as an LP with the replacement of the maximum coordinate of a vector:

n+1 переменных

2n ограничений.

$$
\begin{aligned}
&\min_{t \in \mathbb{R}, x \in \mathbb{R}^n} \quad t \\
\text{s.t. } & \underline{a_i^T x - b_i \leq t}, \; i = 1, \dots, n \\
& \underline{-a_i^T x + b_i \leq t}, \; i = 1, \dots, n
\end{aligned}
$$

$-t_i \leq a_i^T x - b_i \leq t_i$

$t_i = |a_i^T x - b_i|$

$\max_{i \in [1;m]} t_i = t$

$-t \leq a_i^T x - b_i \leq t$

ℓ_1 approximation problem

$$|a_i^T x - b_i| = t_i$$

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_1 \leftrightarrow \min_{x \in \mathbb{R}^n} \sum_{i=1}^n |a_i^T x - b_i| = \min \sum_{i=1}^n t_i$$

Could be equivalently written as an LP with the replacement of the sum of coordinates of a vector:

ℓ_1 approximation problem

$$1^T t = \sum_{i=1}^{n} t_i \cdot 1$$

$$\min_{x \in \mathbb{R}^n} \|Ax - b\|_1 \leftrightarrow \min_{x \in \mathbb{R}^n} \sum_{i=1}^{n} |a_i^T x - b_i|$$

Could be equivalently written as an LP with the replacement of the sum of coordinates of a vector:

$$\begin{aligned}
&\min_{t \in \mathbb{R}^n, x \in \mathbb{R}^n} \mathbf{1}^T t \\
\text{s.t. } & a_i^T x - b_i \leq t_i, \; i = 1, \ldots, n \\
& -a_i^T x + b_i \leq t_i, \; i = 1, \ldots, n
\end{aligned}$$

$$\boxed{\begin{aligned} &\min 1^T t \\ &x \in \mathbb{R}^n, \\ &t \in \mathbb{R}^n \\ &a_i^T x - b_i = t_i \end{aligned}}$$

n огр. „="

2n переменных

Duality in Linear Programming

Duality

Primal problem:

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^n} c^\top x \\
\text{s.t. } & Ax = b \\
& x_i \geq 0, \; i = 1, \dots, n
\end{aligned}
\tag{1}
$$

Duality

Primal problem:

$$
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&\min_{x \in \mathbb{R}^n} c^\top x \\
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& x_i \geq 0, \; i = 1, \dots, n
\end{aligned}
\tag{1}
$$

KKT for optimal x^*, ν^*, λ^*:

$$
\begin{aligned}
&L(x, \nu, \lambda) = c^T x + \nu^T (Ax - b) - \lambda^T x \\
&-A^T \nu^* + \lambda^* = c \\
&Ax^* = b \\
&x^* \succeq 0 \\
&\lambda^* \succeq 0 \\
&\lambda_i^* x_i^* = 0
\end{aligned}
$$

Duality

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&Ax^* = b \\
&x^* \succeq 0 \\
&\lambda^* \succeq 0 \\
&\lambda_i^* x_i^* = 0
\end{aligned}
$$

Has the following dual:

$$
\begin{aligned}
&\max_{\nu \in \mathbb{R}^m} -b^\top \nu \\
\text{s.t. } & -A^T \nu \preceq c
\end{aligned}
\tag{2}
$$

Find the dual problem to the problem above (it should be the original LP). Also, write down KKT for the dual problem, to ensure, they are identical to the primal KKT.

Strong duality in linear programming

(i) If either problem Equation 1 or Equation 2 has a (finite) solution, then so does the other, and the objective values are equal.

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PROOF. For (i), suppose that Equation 1 has a finite optimal solution x^*. It follows from KKT that there are optimal vectors λ^* and ν^* such that (x^*, ν^*, λ^*) satisfies KKT. We noted above that KKT for Equation 1 and Equation 2 are equivalent. Moreover, $c^T x^* = (-A^T\nu^* + \lambda^*)^T x^* = -(\nu^*)^T A x^* = -b^T\nu^*$, as claimed.

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To prove (ii), suppose that the primal is unbounded, that is, there is a sequence of points x_k, $k = 1, 2, 3, \ldots$ such that

$$c^T x_k \downarrow -\infty, \quad Ax_k = b, \quad x_k \geq 0.$$

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$$c^T x_k \downarrow -\infty, \quad Ax_k = b, \quad x_k \geq 0.$$

Suppose too that the dual Equation 2 is feasible, that is, there exists a vector $\bar{\nu}$ such that $-A^T \bar{\nu} \leq c$. From the latter inequality together with $x_k \geq 0$, we have that $-\bar{\nu}^T A x_k \leq c^T x_k$, and therefore

$$-\bar{\nu}^T b = -\bar{\nu}^T A x_k \leq c^T x_k \downarrow -\infty,$$

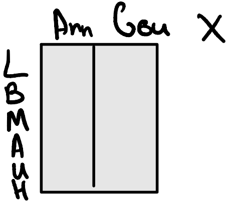
yielding a contradiction. Hence, the dual must be infeasible. A similar argument can be used to show that the unboundedness of the dual implies the infeasibility of the primal.

Example: Transportation problem

The prototypical transportation problem deals with the distribution of a commodity from a set of sources to a set of destinations. The object is to minimize total transportation costs while satisfying constraints on the supplies available at each of the sources, and satisfying demand requirements at each of the destinations.

Figure 1: Western Europe Map. Open In Colab

Example: Transportation problem



Customer / Source	Arnhem [€/ton]	Gouda [€/ton]	Demand [tons]
London	n/a 1000	2.5	125
Berlin	2.5	n/a 1000	175
Maastricht	1.6	2.0	225
Amsterdam	1.4	1.0	250
Utrecht	0.8	1.0	225
The Hague	1.4	0.8	200
Supply [tons]	550 tons	700 tons	

$$\text{minimize:} \quad \text{Cost} = \sum_{c \in \text{Customers}} \sum_{s \in \text{Sources}} T[c,s]x[c,s]$$

$$= \sum_{ij} T_{ij} \cdot X_{ij}$$

тариф перевозок

кол-во перевозок

Example: Transportation problem

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$$\text{minimize:} \quad \text{Cost} = \sum_{c \in \text{Customers}} \sum_{s \in \text{Sources}} T[c,s]x[c,s]$$

$$\sum_{c \in \text{Customers}} x[c,s] \leq \text{Supply}[s] \qquad \forall s \in \text{Sources}$$

Множители Лагранжа

$f_0(x) \to \min\limits_{x \in \mathbb{R}^n}$

$f_i(x) \leq 0$

$L(x, \lambda) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x)$

$\lambda_i = 0 \; ; \; f_i(x) < 0$

ограничение неактивно

$\lambda_i > 0 \; ; \; f_i(x) = 0$

ограничение активно

Example: Transportation problem

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This can be represented in the following graph:

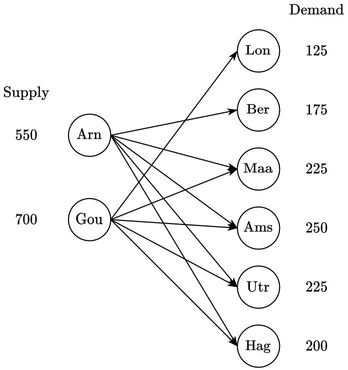


Figure 2: Graph associated with the problem

$$\text{minimize:} \quad \text{Cost} = \sum_{c \in \text{Customers}} \sum_{s \in \text{Sources}} T[c, s] x[c, s]$$

возможности поставщиков не нарушились

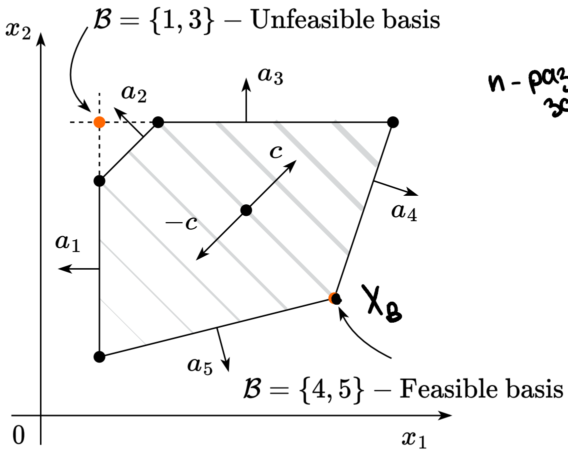
$$\sum_{c \in \text{Customers}} x[c, s] \leq \text{Supply}[s] \qquad \forall s \in \text{Sources}$$

потребности потребителя покрываются

$$\sum_{s \in \text{Sources}} x[c, s] = \text{Demand}[c] \qquad \forall c \in \text{Customers}$$

Simplex Algorithm

Geometry of simplex algorithm

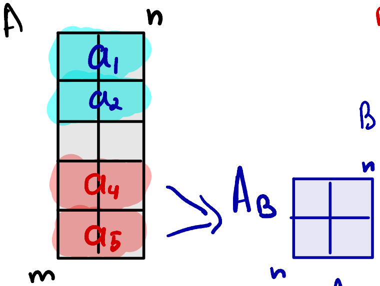


We will consider the following simple formulation of LP, which is, in fact, dual to the Standard form:

n - размерность задачи

$$
\boxed{\begin{aligned} &\min_{x \in \mathbb{R}^n} c^\top x \\ &\text{s.t. } Ax \leq b \end{aligned}} \tag{LP.Inequality}
$$

- Definition: a **basis** $\mathcal{B}$ is a subset of n (integer) numbers between 1 and m, so that $\text{rank} A_{\mathcal{B}} = n$.



$B = \{4, 5\}$ базис ↑ допустимый

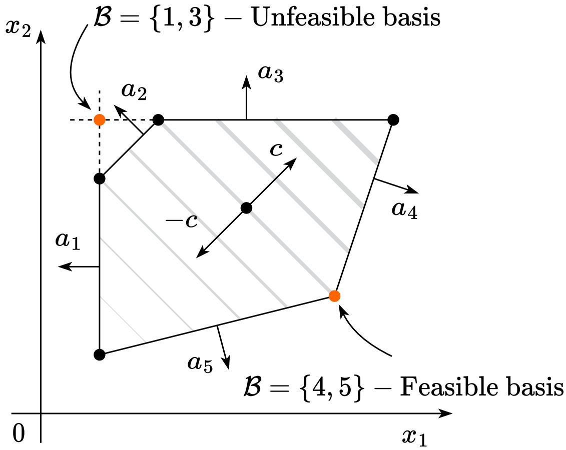
$B = \{1, 3\}$ - недопустимый базис

X_B угл. точка базиса B

$$\underset{n \times n}{A_B} \cdot \underset{n\times 1}{X_B} = \underset{n \times 1}{b_B}$$

$$X_B = A_B^{-1} \cdot b_B$$

Geometry of simplex algorithm



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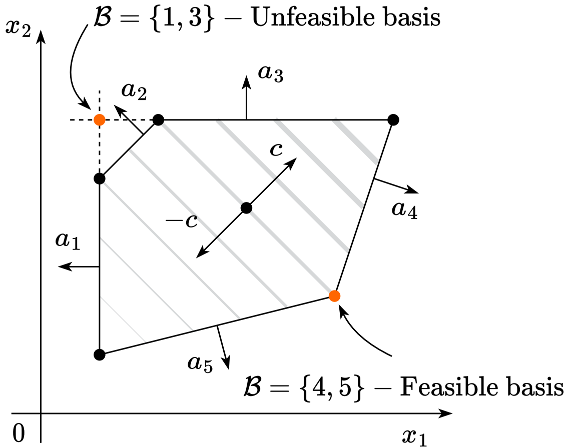
Geometry of simplex algorithm



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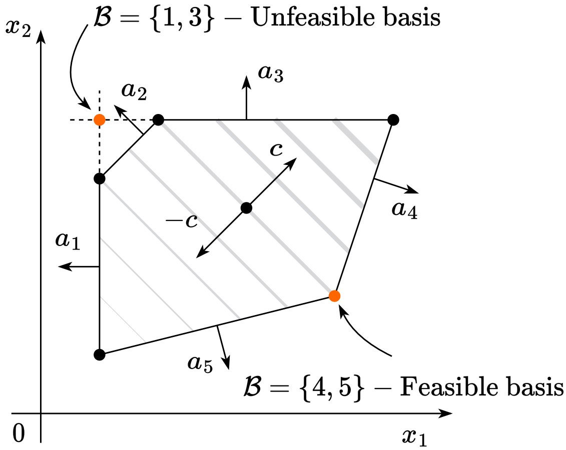
- Also, we can derive a point of intersection of all these hyperplanes from the basis: $x_{\mathcal{B}} = A^{-1}_{\mathcal{B}} b_{\mathcal{B}}$.

Geometry of simplex algorithm



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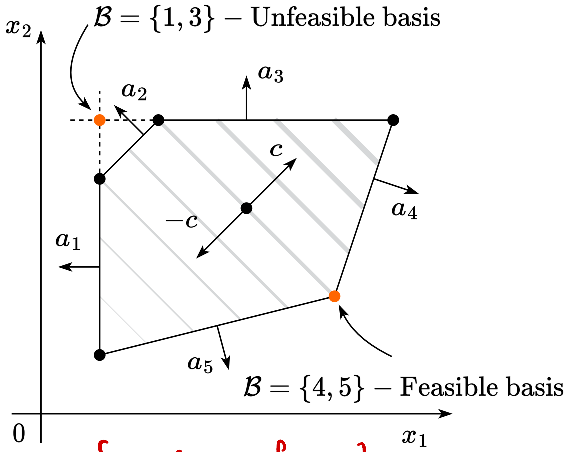
- Also, we can derive a point of intersection of all these hyperplanes from the basis: $x_{\mathcal{B}} = A^{-1}_{\mathcal{B}} b_{\mathcal{B}}$.
- If $Ax_{\mathcal{B}} \leq b$, then basis $\mathcal{B}$ is **feasible**.

$\underset{m \times n}{A} \cdot \underset{n \times 1}{x_B} \leq \underset{m \times 1}{b}$ $\qquad x_B = A_B^{-1} \cdot b_B$

Geometry of simplex algorithm



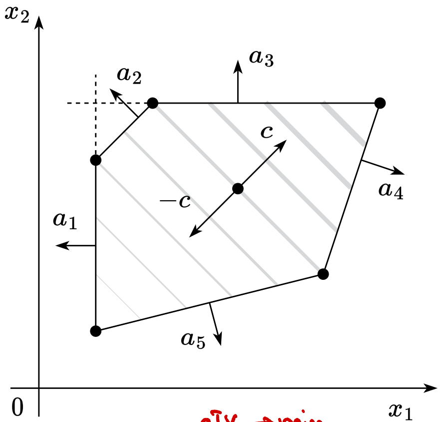
базис $B = \{1, 5\}$ - оптимальный

We will consider the following simple formulation of LP, which is, in fact, dual to the Standard form:

$$\min_{x \in \mathbb{R}^n} c^\top x \qquad \text{s.t. } Ax \leq b \tag{LP.Inequality}$$

- Definition: a **basis** $\mathcal{B}$ is a subset of n (integer) numbers between 1 and m, so that $\text{rank} A_{\mathcal{B}} = n$.
- Note, that we can associate submatrix $A_{\mathcal{B}}$ and corresponding right-hand side $b_{\mathcal{B}}$ with the basis $\mathcal{B}$.
- Also, we can derive a point of intersection of all these hyperplanes from the basis: $x_{\mathcal{B}} = A_{\mathcal{B}}^{-1} b_{\mathcal{B}}$.
- If $Ax_{\mathcal{B}} \leq b$, then basis $\mathcal{B}$ is **feasible**.
- A basis $\mathcal{B}$ is optimal if $x_{\mathcal{B}}$ is an optimum of the LP.Inequality.

The solution of LP if exists lies in the corner



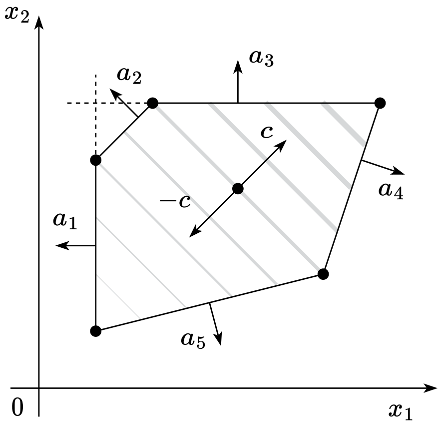
Standard LP
$c^T x \to \min_{x \in \mathbb{R}^n}$
$Ax = b$
$x \geq 0$

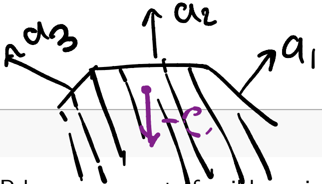
Theorem

1. If Standard LP has a nonempty feasible region, then there is at least one basic feasible point

The high-level idea of the simplex method is following:

The solution of LP if exists lies in the corner

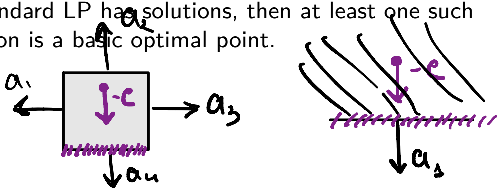




Theorem

1. If Standard LP has a nonempty feasible region, then there is at least one basic feasible point
2. If Standard LP has solutions, then at least one such solution is a basic optimal point.

$m < n$



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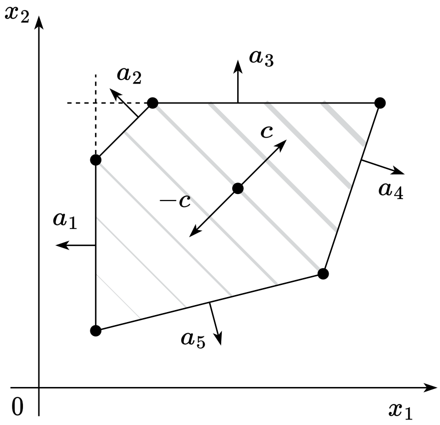

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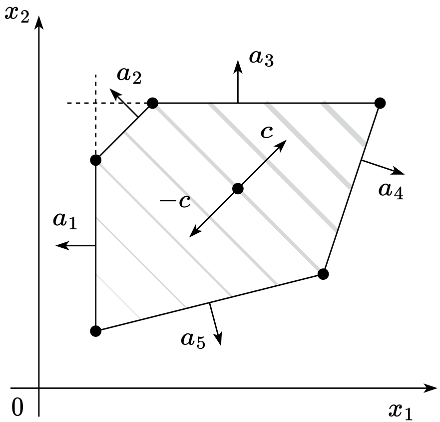

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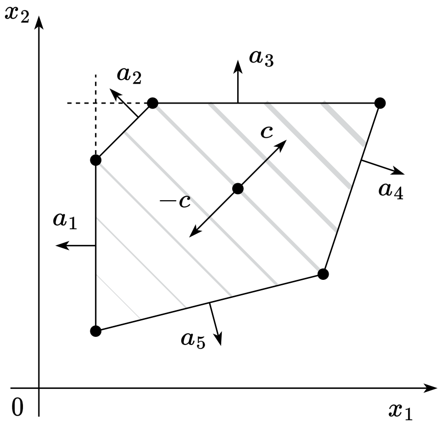

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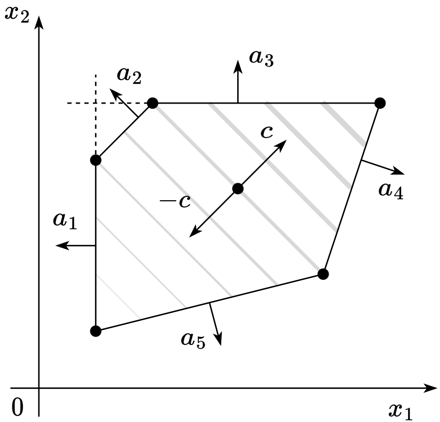
For proof see Numerical Optimization by Jorge Nocedal and Stephen J. Wright theorem 13.2

The high-level idea of the simplex method is following:

- Ensure, that you are in the corner.

The solution of LP if exists lies in the corner



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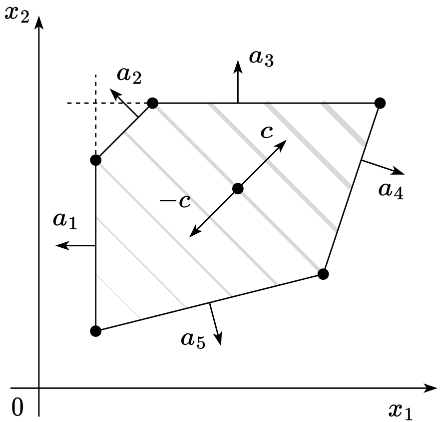
- Check optimality.

The solution of LP if exists lies in the corner



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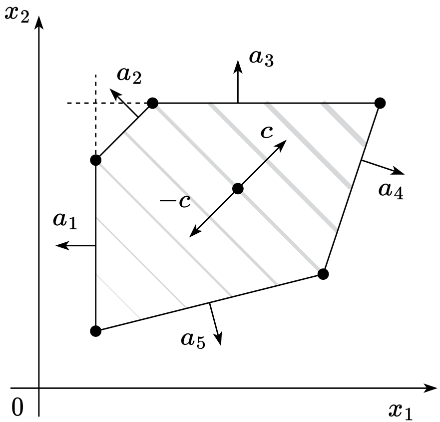
The solution of LP if exists lies in the corner



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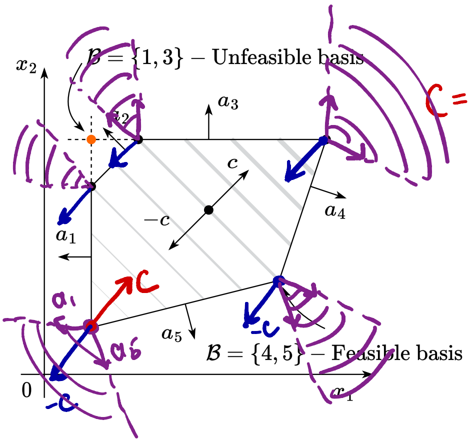
The high-level idea of the simplex method is following:

- Ensure, that you are in the corner.
- Check optimality.
- If necessary, switch the corner (change the basis).
- Repeat until converge.

Optimal basis



Since we have a basis, we can decompose our objective vector c in this basis and find the scalar coefficients $\lambda_{\mathcal{B}}$:

$C = \lambda_1 a_1 + \ldots + \lambda_n a_n$

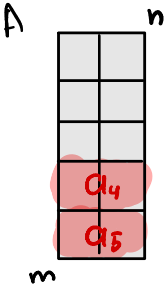
$$\boxed{\lambda_{\mathcal{B}}^T A_{\mathcal{B}} = c^T} \leftrightarrow \lambda_{\mathcal{B}}^T = c^T A_{\mathcal{B}}^{-1}$$

Theorem
- $\lambda_1 \ldots \lambda_n \leq 0 \Rightarrow B$ – оптимальный
- B – допустимый

If all components of $\lambda_{\mathcal{B}}$ are non-positive and $\mathcal{B}$ is feasible, then $\mathcal{B}$ is optimal.

$\lambda_B A_B = -C$

Proof



$$\exists x^* : Ax^* \leq b, c^T x^* < c^T x_{\mathcal{B}}$$

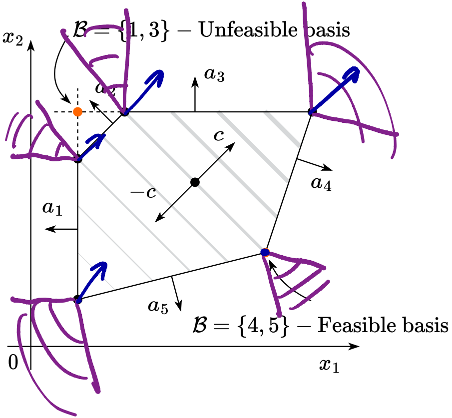
A_B $\qquad$ b_B

$\lambda_B^T = c^T A_B^{-1}$

$$\boxed{\lambda_B = (A_B^{-1})^T \cdot c}$$

≤ 0

Optimal basis



Since we have a basis, we can decompose our objective vector c in this basis and find the scalar coefficients $\lambda_{\mathcal{B}}$:

$$\lambda_{\mathcal{B}}^T A_{\mathcal{B}} = c^T \leftrightarrow \lambda_{\mathcal{B}}^T = c^T A_{\mathcal{B}}^{-1}$$

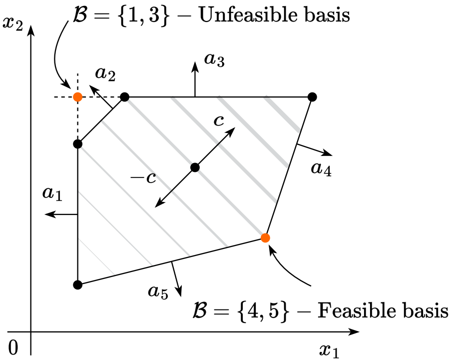
Theorem

If all components of $\lambda_{\mathcal{B}}$ are non-positive and $\mathcal{B}$ is feasible, then $\mathcal{B}$ is optimal.

Proof

$$\exists x^* : Ax^* \leq b, c^T x^* < c^T x_{\mathcal{B}}$$
$$A_{\mathcal{B}} x^* \leq b_{\mathcal{B}}$$

Optimal basis



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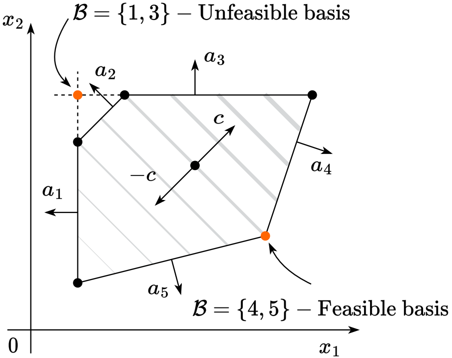
Proof

$$\exists x^* : Ax^* \leq b, c^T x^* < c^T x_\mathcal{B}$$

$$A_\mathcal{B} x^* \leq b_\mathcal{B}$$

$$\lambda_\mathcal{B}^T A_\mathcal{B} x^* \geq \lambda_\mathcal{B}^T b_\mathcal{B}$$

Optimal basis



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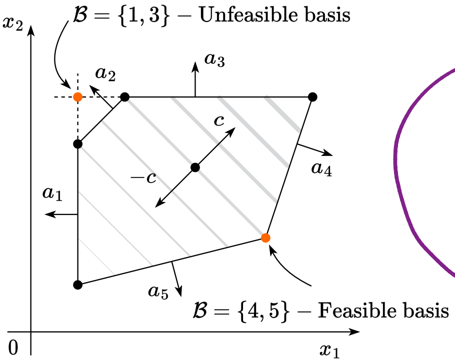
$$c^T x^* \geq \lambda_{\mathcal{B}}^T A_{\mathcal{B}} x_{\mathcal{B}}$$

Optimal basis



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Proof пусть , но $x_{\mathcal{B}}$ – не оптим., т.е. $\exists x^*$: $c^T x^* \leq c^T x_{\mathcal{B}}$

$A x^* \leq b$

$\lambda_{\mathcal{B}} \leq 0$

$$\exists x^* : Ax^* \leq b, c^T x^* < c^T x_{\mathcal{B}}$$

$$A_{\mathcal{B}} x^* \leq b_{\mathcal{B}} \quad | \lambda_{\mathcal{B}}^T \cdot$$

$$\lambda_{\mathcal{B}}^T A_{\mathcal{B}} x^* \geq \lambda_{\mathcal{B}}^T b_{\mathcal{B}}$$

c^T

$$c^T x^* \geq \lambda_{\mathcal{B}}^T A_{\mathcal{B}} x_{\mathcal{B}}$$

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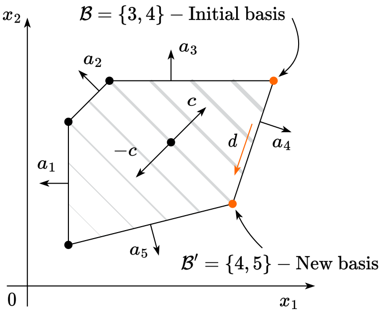
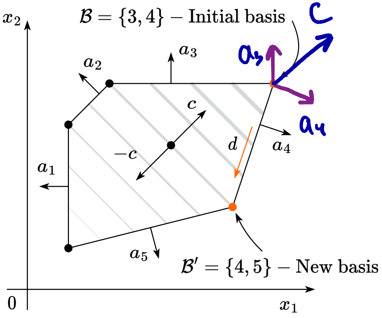
Changing basis



Suppose, some of the coefficients of $\lambda_{\mathcal{B}}$ are positive. Then we need to go through the edge of the polytope to the new vertex (i.e., switch the basis)

- Suppose, we have a basis $\mathcal{B}$: $\lambda_{\mathcal{B}}^T = c^T A_{\mathcal{B}}^{-1}$

Changing basis



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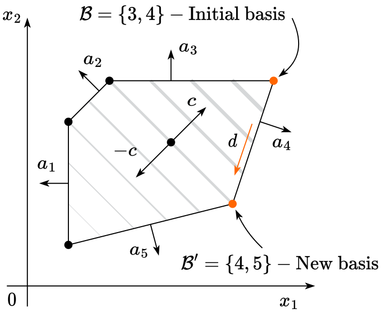
Changing basis



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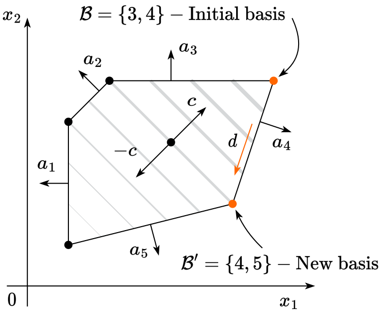
$$\begin{matrix} n\text{-}1 \text{ ур.} \\ 1 \text{ ур.} \end{matrix} \begin{cases} A_{\mathcal{B} \setminus \{k\}} d = 0 \\ a_k^T d = -1 \end{cases}$$

не нарушаем остальные ограничения

двигаемся в сторону уменьш. знач. $f(x)$

Changing basis



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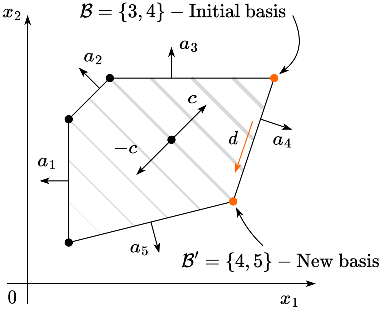
$$\begin{cases} A_{\mathcal{B}\setminus\{k\}} d = 0 \\ a_k^T d = -1 \end{cases} \qquad c^T d$$

Changing basis



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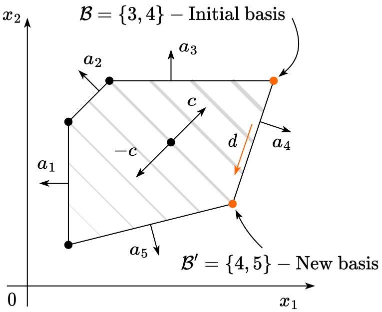
Changing basis



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как ведёт себя $f(x)$

$$
\begin{cases} A_{\mathcal{B} \setminus \{k\}} d = 0 \\ a_k^T d = -1 \end{cases}
\qquad
\underline{c^T d} = \lambda^T_{\mathcal{B}} A_{\mathcal{B}} d = \sum_{i=1}^{n} \lambda^i_{\mathcal{B}} (A_{\mathcal{B}} d)^i
$$

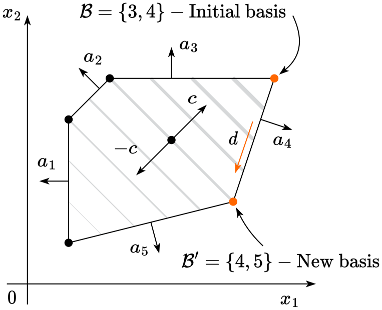
$$x \rightsquigarrow x + \mu d$$

$$f(x_{new}) = f(x_{old} + \mu d) =$$

$$= c^T(x_{old} + \mu d) =$$

$$= c^T x_{old} + \mu c^T d$$

Changing basis



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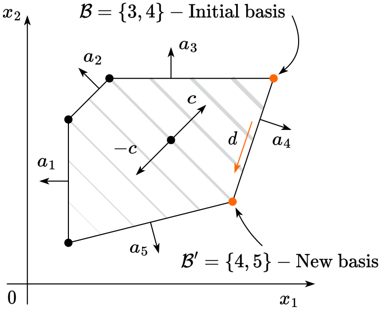
Changing basis



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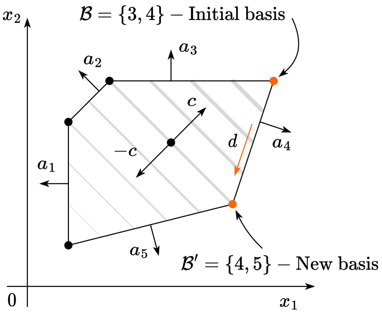
$\{3,4\}$ $j = \{1,2,5\}$

- For all $j \notin \mathcal{B}$ calculate the projection stepsize:

$$\mu_j = \frac{b_j - a_j^T x_{\mathcal{B}}}{a_j^T d}$$

μ_1, μ_2, μ_5

Changing basis



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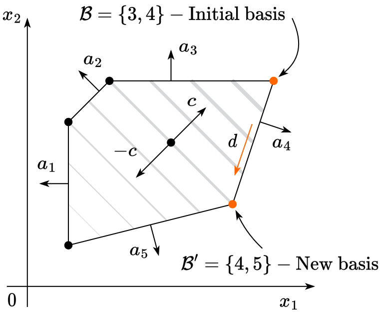
- Define the new vertex, that you will add to the new basis:

$$t = \text{arg}\min_j \{\mu_j \mid \mu_j > 0\}$$

$$\mathcal{B}' = \mathcal{B} \setminus \{k\} \cup \{t\}$$

$$x_{\mathcal{B}'} = x_\mathcal{B} + \mu_t d = A^{-1}_{\mathcal{B}'} b_{\mathcal{B}'}$$

Changing basis



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$$x_{\mathcal{B}'} = x_\mathcal{B} + \mu_t d = A^{-1}_{\mathcal{B}'} b_{\mathcal{B}'}$$

- Note, that changing basis implies objective function decreasing

$$c^T x_{\mathcal{B}'} = c^T (x_\mathcal{B} + \mu_t d) = c^T x_\mathcal{B} + \mu_t c^T d$$

Finding an initial basic feasible solution

рандомный выбор - плохо

все наборы длины n из m вариантов

$$C_m^n = \frac{m!}{n!(m-n)!}$$

We aim to solve the following problem:

$$\begin{split}
&\min_{x \in \mathbb{R}^n} c^\top x \\
\text{s.t. } & Ax \leq b
\end{split} \tag{3}$$

The proposed algorithm requires an initial basic feasible solution and corresponding basis.

СХЕМА: Вместо исх. задачи (P) решаем вспомогательную, стартовая (угловая) точка для которой легко записывается, а решение всп. явл. старт. точкой (P) (угловой)

Finding an initial basic feasible solution

$x = y - z$

We aim to solve the following problem:

$$
\begin{aligned}
\min_{x \in \mathbb{R}^n} \; & c^\top x \\
\text{s.t. } & Ax \leq b
\end{aligned}
\tag{3}
$$

n мерная

$\Longleftrightarrow$

We start by reformulating the problem:

$$
\begin{aligned}
\min_{y \in \mathbb{R}^n, z \in \mathbb{R}^n} \; & c^\top (y - z) \\
\text{s.t. } & Ay - Az \leq b \\
& y \geq 0, z \geq 0
\end{aligned}
\tag{4}
$$

2n мерная

The proposed algorithm requires an initial basic feasible solution and corresponding basis.

Finding an initial basic feasible solution

$x = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ $\quad x = y - z \quad y = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad z = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$

We aim to solve the following problem:

$$\begin{aligned} &\min_{x \in \mathbb{R}^n} c^\top x \\ &\text{s.t. } Ax \leq b \end{aligned} \tag{3}$$

We start by reformulating the problem:

$$\begin{aligned} &\min_{y \in \mathbb{R}^n, z \in \mathbb{R}^n} c^\top (y - z) \\ &\text{s.t. } Ay - Az \leq b \\ &y \geq 0, z \geq 0 \end{aligned} \tag{4}$$

The proposed algorithm requires an initial basic feasible solution and corresponding basis.

Given the solution of Problem 4 the solution of Problem 3 can be recovered and vice versa

$$x = y - z \qquad \Leftrightarrow \qquad y_i = \max(x_i, 0), \quad z_i = \max(-x_i, 0)$$

Now we will try to formulate new LP problem, which solution will be basic feasible point for Problem 4. Which means, that we firstly run Simplex algorithm for Phase-1 problem and run Phase-2 problem with known starting point. Note, that basic feasible solution for Phase-1 should be somehow easily established.

Finding an initial basic feasible solution

$$
\begin{aligned}
\min_{y \in \mathbb{R}^n, z \in \mathbb{R}^n} \ & c^\top (y - z) \\
\text{s.t. } & Ay - Az \leq b \\
& y \geq 0, z \geq 0
\end{aligned}
\qquad \text{(Phase-2 (Main LP))}
$$

Finding an initial basic feasible solution

$A \in \mathbb{R}^{m \times n}$

$$
\begin{aligned}
\min_{y \in \mathbb{R}^n, z \in \mathbb{R}^n} \ & c^\top (y - z) \\
\text{s.t. } & Ay - Az \leq b \\
& y \geq 0, z \geq 0
\end{aligned}
\tag{Phase-2 (Main LP)}
$$

SLACK

(переменные дополн. нежест.)

ВСПОМОГАТЕЛЬНАЯ ЗАДАЧА

$$
\begin{aligned}
\min_{\xi \in \mathbb{R}^m, y \in \mathbb{R}^n, z \in \mathbb{R}^n} \ & \sum_{i=1}^m \xi_i \\
\text{s.t. } & Ay - Az \leq b + \xi \\
& y \geq 0, z \geq 0, \xi \geq 0
\end{aligned}
\tag{Phase-1}
$$

Finding an initial basic feasible solution

$$
\begin{aligned}
\min_{y \in \mathbb{R}^n, z \in \mathbb{R}^n} \; & c^\top (y - z) \\
\text{s.t. } & Ay - Az \leq b \\
& y \geq 0, z \geq 0
\end{aligned}
\qquad \text{(Phase-2 (Main LP))}
$$

$$
\begin{aligned}
\min_{\xi \in \mathbb{R}^m, y \in \mathbb{R}^n, z \in \mathbb{R}^n} \; & \sum_{i=1}^{m} \xi_i = 0 \quad (\xi_i = 0) \\
\text{s.t. } & Ay - Az \leq b + \xi \\
& y \geq 0, z \geq 0, \xi \geq 0
\end{aligned}
\qquad \text{(Phase-1)}
$$

- If Phase-2 (Main LP) problem has a feasible solution, then Phase-1 optimum is zero (i.e. all slacks ξ_i are zero).
  Proof: trivial check.

Пусть $\exists\, y, z$:

$$Ay - Az \leq b + \overset{=0}{\xi}, \quad \xi \geq 0$$

$$y \geq 0, \; z \geq 0$$

Finding an initial basic feasible solution

угловая точка
= выполнены

$$\min_{y\in\mathbb{R}^n, z\in\mathbb{R}^n} c^\top(y-z)$$

2n

$$\text{s.t. } Ay - Az \le b$$

(Phase-2 (Main LP))

как равенства

$$y \ge 0, z \ge 0$$

2n огранич.

$$\min_{\xi\in\mathbb{R}^m, y\in\mathbb{R}^n, z\in\mathbb{R}^n} \sum_{i=1}^m \xi_i$$

(Phase-1)

$$\text{s.t. } Ay - Az \le b + \xi$$

m нер-в

$$y \ge 0, z \ge 0, \xi \ge 0$$

пусть $\underbrace{\xi^*}_{0}, y^*, z^*$

$Ay^* - Az^* \le b$

$y^* \ge 0 \quad z^* \ge 0$

- If Phase-2 (Main LP) problem has a feasible solution, then Phase-1 optimum is zero (i.e. all slacks ξ_i are zero).
  Proof: trivial check.
- If Phase-1 optimum is zero (i.e. all slacks ξ_i are zero), then we get a feasible basis for Phase-2.
  Proof: trivial check.

оптим. решение Phase-1 $\Rightarrow$ старт, углов. точка Phase-2

как найти угл. точку?

Finding an initial basic feasible solution

$$
\begin{aligned}
&\min_{y \in \mathbb{R}^n, z \in \mathbb{R}^n} c^\top (y - z) \\
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&\min_{\xi \in \mathbb{R}^m, y \in \mathbb{R}^n, z \in \mathbb{R}^n} \sum_{i=1}^{m} \xi_i \\
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Finding an initial basic feasible solution

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\text{s.t. } & Ay - Az \leq b \\
& y \geq 0, z \geq 0
\end{aligned}
\tag{Phase-2 (Main LP)}
$$

2n переменных

$$
\begin{aligned}
&\min_{\xi\in\mathbb{R}^m, y\in\mathbb{R}^n, z\in\mathbb{R}^n} \sum_{i=1}^m \xi_i \\
\text{s.t. } & \underline{Ay - Az \leq b + \xi} \\
& \underline{y \geq 0, z \geq 0, \xi \geq 0}
\end{aligned}
\tag{Phase-1}
$$

2n+m огр.
как раз

- If Phase-2 (Main LP) problem has a feasible solution, then Phase-1 optimum is zero (i.e. all slacks ξ_i are zero).
  Proof: trivial check.
- If Phase-1 optimum is zero (i.e. all slacks ξ_i are zero), then we get a feasible basis for Phase-2.
  Proof: trivial check.

пусть мы решили Phase 1

$y^*, z^*, 0 \leftarrow$ угл. точка Phase 1

- Now we know, that if we can solve a Phase-1 problem then we will either find a starting point for the simplex method in the original method (if slacks are zero) or verify that the original problem was infeasible (if slacks are non-zero).

$Ay^* - Az^* \leq b$

$y^* \geq 0, z^* \geq 0$

Finding an initial basic feasible solution

$$
\begin{aligned}
&\min_{y \in \mathbb{R}^n, z \in \mathbb{R}^n} c^\top (y - z) \\
\text{s.t. } & Ay - Az \leq b \\
& y \geq 0, z \geq 0
\end{aligned}
\qquad \text{(Phase-2 (Main LP))}
$$

2n+m мерная

$$
\begin{aligned}
&\min_{\xi \in \mathbb{R}^m, y \in \mathbb{R}^n, z \in \mathbb{R}^n} \sum_{i=1}^m \xi_i \\
\text{s.t. } & Ay - Az \leq b + \xi \\
& y \geq 0, z \geq 0, \xi \geq 0
\end{aligned}
\qquad \text{(Phase-1)}
$$

$y = 0$ $z = 0$ | 2n штук

- If Phase-2 (Main LP) problem has a feasible solution, then Phase-1 optimum is zero (i.e. all slacks ξ_i are zero).
  Proof: trivial check.
- If Phase-1 optimum is zero (i.e. all slacks ξ_i are zero), then we get a feasible basis for Phase-2.
  Proof: trivial check.

$$
\begin{array}{cccc}
 & b & & \\
0 \leq & 1 & + & 0 \\
0 \leq & -2 & + & 2 \\
0 \leq & 3 & + & 0
\end{array}
$$

– усл. (вып. как рав.)

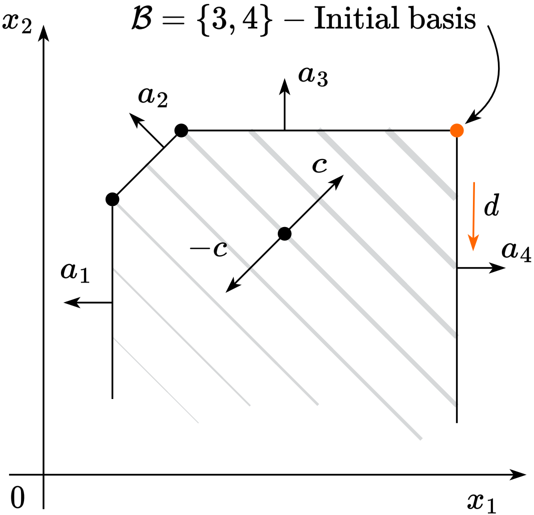
- Now we know, that if we can solve a Phase-1 problem then we will either find a starting point for the simplex method in the original method (if slacks are zero) or verify that the original problem was infeasible (if slacks are non-zero).
- But how to solve Phase-1? It has basic feasible solution (the problem has $2n + m$ variables and the point below ensures $2n + m$ inequalities are satisfied as equalities (active).)

$$
\boxed{\underbrace{z = 0}_{n} \quad \underbrace{y = 0}_{n} \quad \underbrace{\xi_i = \max(0, -b_i)}_{m}}
$$

2n+m ограничений выполнены как равенства

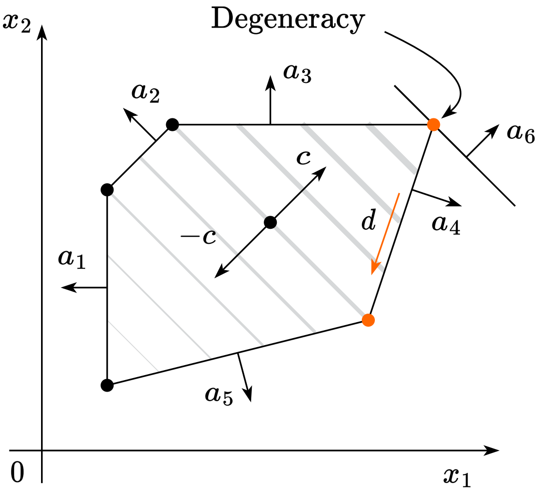
Convergence of the Simplex Algorithm

Unbounded budget set



In this case, all μ_j will be negative.

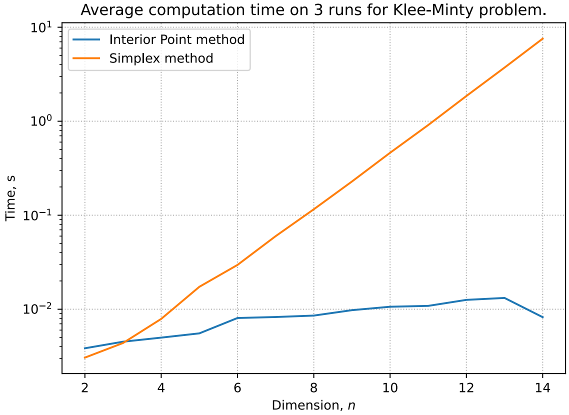
Degeneracy



One needs to handle degenerate corners carefully. If no degeneracy exists, one can guarantee a monotonic decrease of the objective function on each iteration.

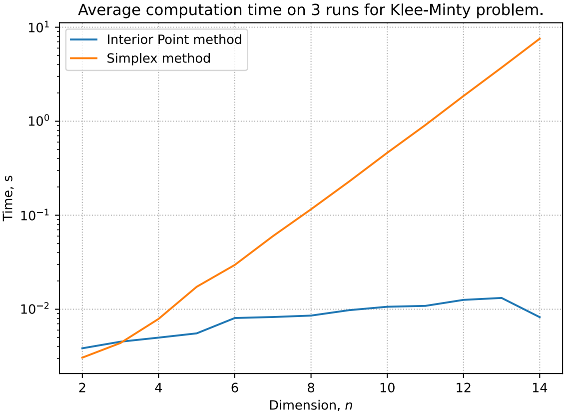
$\mu = 0$

Exponential convergence



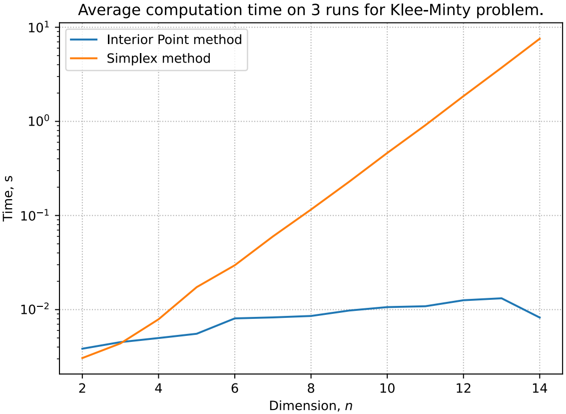
- A wide variety of applications could be formulated as linear programming.

Exponential convergence



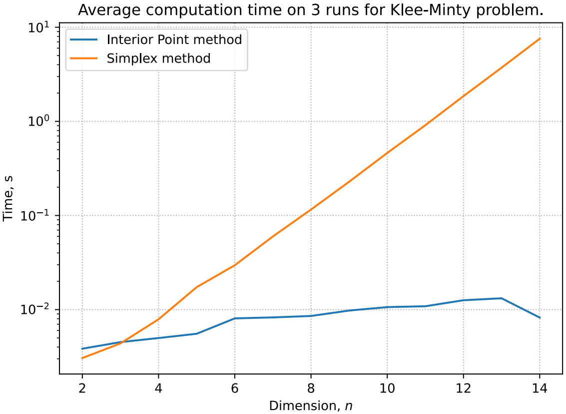
- A wide variety of applications could be formulated as linear programming.
- Simplex algorithm is simple but could work exponentially long.

Exponential convergence



- A wide variety of applications could be formulated as linear programming.
- Simplex algorithm is simple but could work exponentially long.
- Khachiyan's ellipsoid method (1979) is the first to be proven to run at polynomial complexity for LPs. However, it is usually slower than simplex in real problems.

Exponential convergence



- A wide variety of applications could be formulated as linear programming.
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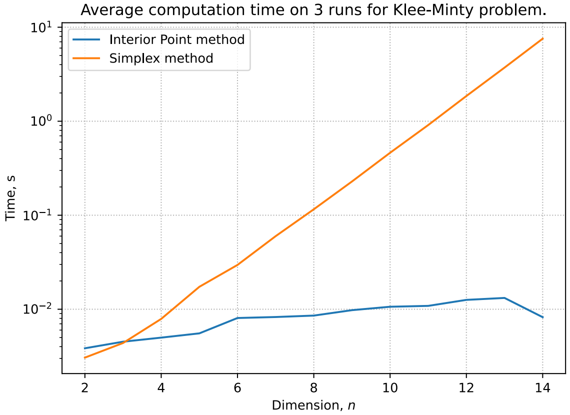
- Major breakthrough - Narendra Karmarkar's method for solving LP (1984) using interior point method.

Exponential convergence



- A wide variety of applications could be formulated as linear programming.
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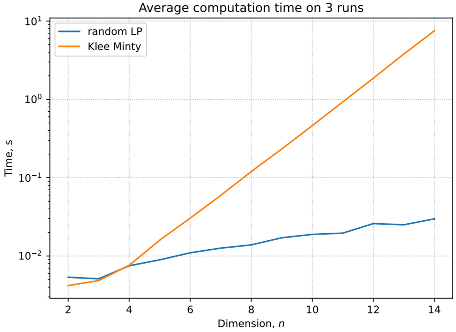
- Major breakthrough - Narendra Karmarkar's method for solving LP (1984) using interior point method.
- Interior point methods are the last word in this area. However, good implementations of simplex-based methods and interior point methods are similar for routine applications of linear programming.

Klee Minty example

Since the number of edge points is finite, the algorithm should converge (except for some degenerate cases, which are not covered here). However, the convergence could be exponentially slow, due to the high number of edges. There is the following iconic example when the simplex algorithm should perform exactly all vertexes.

In the following problem, the simplex algorithm needs to check $2^n - 1$ vertexes with $x_0 = 0$.

$$
\begin{aligned}
& \max_{x \in \mathbb{R}^n} 2^{n-1}x_1 + 2^{n-2}x_2 + \dots + 2x_{n-1} + x_n \\
\text{s.t. } & x_1 \leq 5 \\
& 4x_1 + x_2 \leq 25 \\
& 8x_1 + 4x_2 + x_3 \leq 125 \\
& \ldots \\
& 2^n x_1 + 2^{n-1}x_2 + 2^{n-2}x_3 + \ldots + x_n \leq 5^n \\
& x \geq 0
\end{aligned}
$$



Other

Minimization of convex function as LP

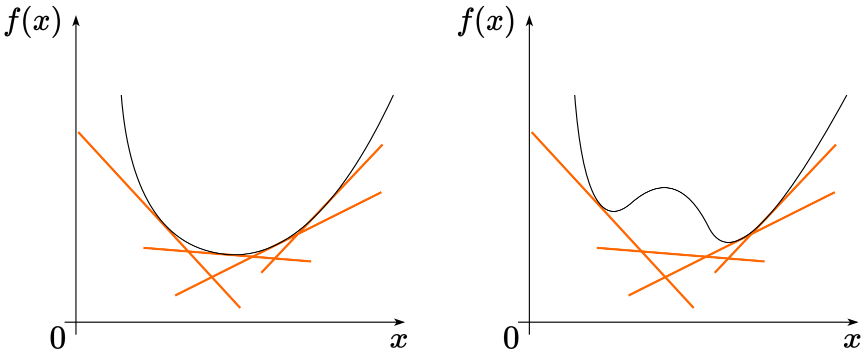


Figure 3: How LP can help with general convex problem

- The function is convex iff it can be represented as a pointwise maximum of linear functions.

Minimization of convex function as LP

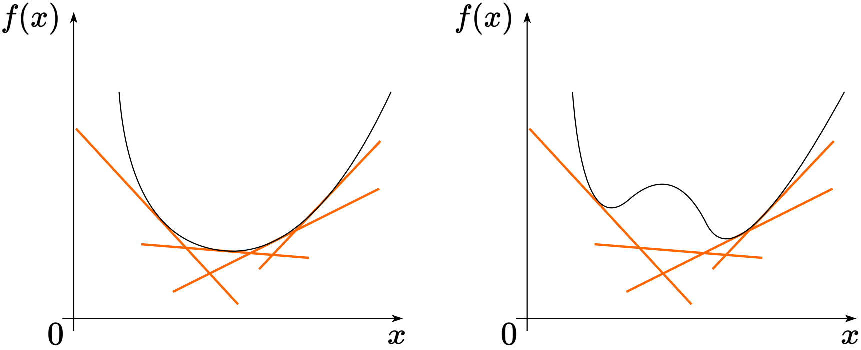


Figure 3: How LP can help with general convex problem

- The function is convex iff it can be represented as a pointwise maximum of linear functions.
- In high dimensions, the approximation may require too many functions.

Minimization of convex function as LP

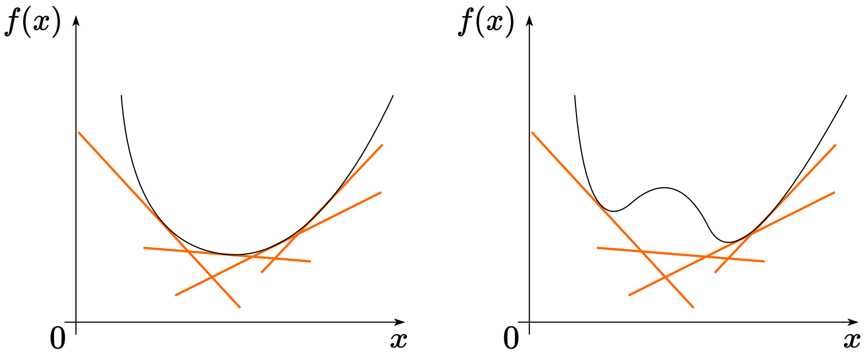


Figure 3: How LP can help with general convex problem

- The function is convex iff it can be represented as a pointwise maximum of linear functions.
- In high dimensions, the approximation may require too many functions.
- More efficient convex optimizers (not reducing to LP) exist.

Mixed Integer Programming

Complexity of MIP

Consider the following Mixed Integer Programming (MIP):

$$
\begin{aligned}
z = 8x_1 + 11x_2 + 6x_3 + 4x_4 \rightarrow \max_{x_1, x_2, x_3, x_4} \\
\text{s.t. } 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \\
x_i \in \{0, 1\} \quad \forall i
\end{aligned}
\tag{5}
$$

Complexity of MIP

LP - релаксация
выпуклая

Consider the following Mixed Integer Programming (MIP): Relax it to:

$$
\begin{aligned}
z = 8x_1 + 11x_2 + 6x_3 + 4x_4 &\to \max_{x_1,x_2,x_3,x_4} \\
\text{s.t. } 5x_1 + 7x_2 + 4x_3 + 3x_4 &\leq 14 \\
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x_i \in [0,1] \quad &\forall i
\end{aligned}
\tag{6}
$$

Complexity of MIP

Consider the following Mixed Integer Programming (MIP):

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z = 8x_1 + 11x_2 + 6x_3 + 4x_4 &\to \max_{x_1,x_2,x_3,x_4} \\
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x_i \in \{0,1\} \quad &\forall i
\end{aligned}
\tag{5}
$$

Optimal solution

$$
x_1 = 0, x_2 = x_3 = x_4 = 1, \text{ and } z = 21.
$$

$$
\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad z = 21
$$

Relax it to:

$$
\begin{aligned}
z = 8x_1 + 11x_2 + 6x_3 + 4x_4 &\to \max_{x_1,x_2,x_3,x_4} \\
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x_i \in [0, 1] \quad &\forall i
\end{aligned}
\tag{6}
$$

Optimal solution

$$
x_1 = x_2 = 1, x_3 = 0.5, x_4 = 0, \text{ and } z = 22.
$$

$$
\begin{pmatrix} 1 \\ 1 \\ 0.5 \\ 0 \end{pmatrix} \quad z = 22
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Complexity of MIP

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Optimal solution

$$
x_1 = x_2 = 1, x_3 = 0.5, x_4 = 0, \text{ and } z = 22.
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- Rounding $x_3 = 0$: gives $z = 19$.

Complexity of MIP

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Optimal solution

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- Rounding $x_3 = 0$: gives $z = 19$.
- Rounding $x_3 = 1$: Infeasible.

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

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- Rounding $x_3 = 1$: Infeasible.

! MIP is much harder, than LP

- Naive rounding of LP relaxation of the initial MIP problem might lead to infeasible or suboptimal solution.

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! MIP is much harder, than LP

- Naive rounding of LP relaxation of the initial MIP problem might lead to infeasible or suboptimal solution.
- General MIP is NP-hard.

Complexity of MIP

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x_i \in \{0, 1\} \quad \forall i
\end{aligned}
\tag{5}
$$

Optimal solution

$$
x_1 = 0, x_2 = x_3 = x_4 = 1, \text{ and } z = 21.
$$

Relax it to:

$$
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\end{aligned}
\tag{6}
$$

Optimal solution

$$
x_1 = x_2 = 1, x_3 = 0.5, x_4 = 0, \text{ and } z = 22.
$$

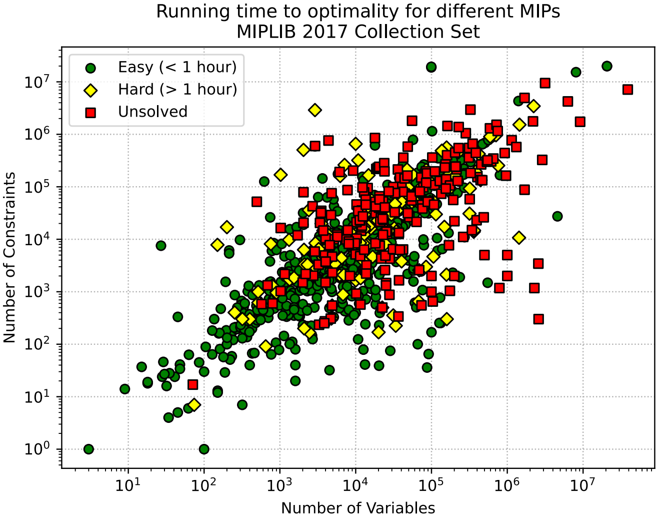
- Rounding $x_3 = 0$: gives $z = 19$.
- Rounding $x_3 = 1$: Infeasible.

! MIP is much harder, than LP

- Naive rounding of LP relaxation of the initial MIP problem might lead to infeasible or suboptimal solution.
- General MIP is NP-hard.
- However, if the coefficient matrix of an MIP is a *totally unimodular matrix*, then it can be solved in polynomial time.

Unpredictable complexity of MIP

- It is hard to predict what will be solved quickly and what will take a long time



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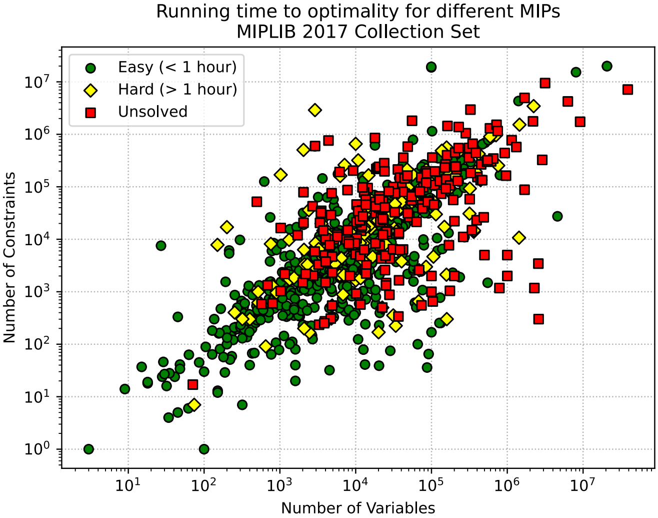

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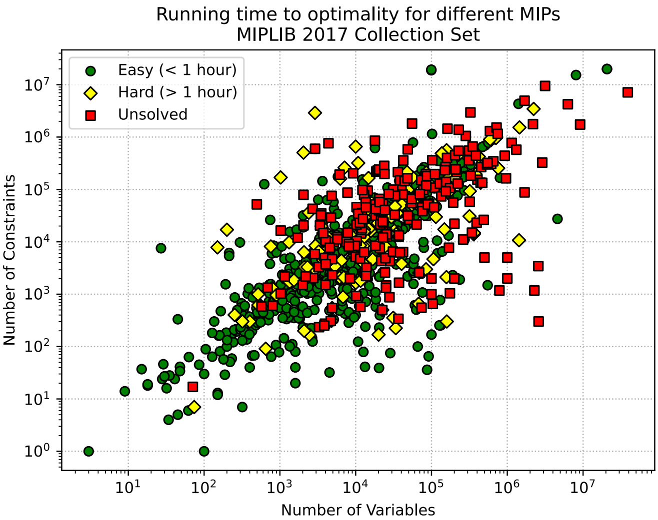

Hardware progress vs Software progress

What would you choose, assuming, that the question posed correctly (you can compile software for any hardware and the problem is the same for both options)? We will consider the time period from 1992 to 2023.

Hardware

Solving MIP with an old software on the modern hardware

Software

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Moore's law states, that computational power doubles every 18 monthes.

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R. Bixby conducted an intensive experiment with benchmarking all CPLEX software version starting from 1992 to 2007 and measured overall software progress (29000 times), later (in 2009) he was a cofounder of Gurobi optimization software, which gives additional ≈ 81 speedup on MILP.

[1] R. Bixby report Recent study

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It turns out that if you need to solve a MILP, it is better to use an old computer and modern methods than vice versa, the newest computer and methods of the early 1990s![1]

[1] R. Bixby report Recent study

Idea of Branch and Bound method

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   - The best known integer solution at the end of the process is the optimal solution to the original MIP.

MIP Example

Consider the following MIP:

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