Markowitz Portfolio Optimization. Optimality Conditions. KKT theorem.

Daniil Merkulov

Applied Math for Data Science. Sberuniversity.

Portfolio optimization

Portfolio optimization

есть n активов $w \in \mathbb{R}^n$

нефть
APL
NVIDIA

$$w = \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{2}{3} \end{pmatrix}$$

$w_i \geq 0$

$\sum_{i=1}^{n} w_i = 1$

$w \succeq 0$

$1^T w = 1$

$w = ?$

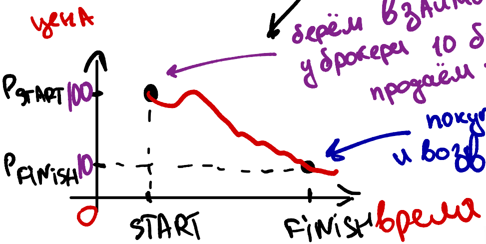
Link to the code

SHORT позиция $w_i < 0$

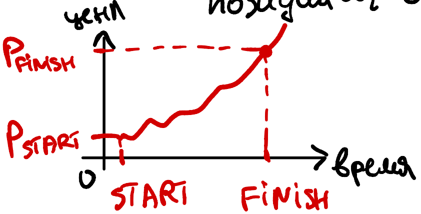
берём взаймы у брокера 10 бумаг; продаём их сразу (1000 $)

покупаем 10 бумаг (100 $) и возвращаем их брокеру.

900 $



LONG позиция $w_i > 0$



Portfolio optimization

Плечо портфеля (leverage) $\quad 1^T w = 1$

$$\|w\|_1 = \sum_{i=1}^{n} |w_i| = |w_1| + |w_2| + \ldots + |w_n|$$

↓ 0

- пусть в портфеле нет SHORT позиций: $w_i \geq 0$

$$\Rightarrow \|w_i\| = \sum_{i=1}^{n} w_i = 1$$

Link to the code

$w_k = -0.1$

- пусть есть SHORT позиция $w_k < 0$: $(+0.1)$

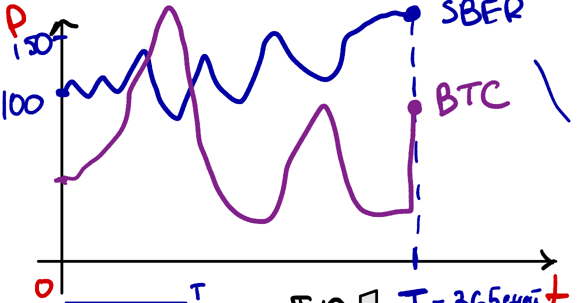
$$\Rightarrow \|w_i\| = \sum_{\substack{i=1 \\ i \neq k}}^{n} w_i + |w_k| = \sum_{\substack{i=1 \\ i \neq k}}^{n} w_i - w_k > 1$$

$\sum_{i=1, i\neq k}^{n} w_i = 1.1$, $\quad 1.1 + 0.1 = 1.2$

чем больше SHORT позиций в портфеле, тем больше $\|w\|_1$.

Portfolio optimization

Историческая статистика



n - число активов в портфеле

1. Считаем

$$r_i = \frac{P_i^+ - P_i}{P_i} \quad r \in \mathbb{R}^n$$

для 1 дня

$$r_{SBER} = \frac{150-100}{100} = 0.5$$

Link to the code

данные: цены T, n → return T, n → $\mu = \mathbb{E} r$, $\mu \in \mathbb{R}^n$

T = 365 дней

portfolio return

$$\sum_{i=1}^{n} w_i \cdot r_i = w^T r$$

2. Считаем матрицу ковариаций:

$$\Sigma \in \mathbb{R}^{n \times n} = \mathbb{E}_r (r-\mu)(r-\mu)^T$$

риск портфеля:

$$R(w) = \underset{1\times n}{w^T} \underset{n\times n}{\Sigma} \underset{n\times 1}{w}$$

Portfolio optimization

максим. ожид. прибыли — мин. риска портфеля — $\mu = \mathbb{E} r$

$$
\begin{array}{ll}
\text{maximize} & \mu^T w - \gamma w^T \Sigma w \\
\text{subject to} & \mathbf{1}^T w = 1, \quad w \in \mathcal{W},
\end{array}
$$

γ — риск-профиль, $\gamma \geq 0$

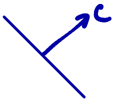
по построению $\Sigma \succeq 0$

$$\min \ \gamma w^T \Sigma w - \mu^T w$$

выпукло — $w \in \mathbb{R}^n$

выпукло — $1^T w = 1$ гиперплоскость

$c^T x = b$



$d^2 f = \langle \nabla^2 f \cdot dx_1, dx \rangle$

$\langle \nabla^2 f \, dx, dx_1 \rangle$

Link to the code

long-only $w_i \geq 0$ +

short welcome $\|w\|_1 \leq 1.1$

1) $f(w) = \gamma w^T \Sigma w - \mu^T w$ — выпукла!

$df = d(\gamma w^T \Sigma w) - d(\mu^T w) = \gamma \cdot d(\langle w, \Sigma w \rangle) - d(\langle \mu, w \rangle)$

$= \gamma \langle 2 \Sigma w, dw \rangle - \langle \mu, dw \rangle \Rightarrow$ $\boxed{\nabla f = 2\gamma \Sigma w - \mu}$

2) $d^2 f = \langle d(2\gamma \Sigma w - \mu), dw_1 \rangle = \langle 2\gamma \Sigma dw, dw_1 \rangle \Rightarrow$ $\boxed{\nabla^2 f = 2\gamma \Sigma}$

Portfolio optimization

мн-во $S = \{w \in \mathbb{R}^n : \|w\|_1 \leq 1.1\}$

S - выпукло?

проверьте по опр.:

$\theta \in [0, 1]$

$w_1 \in S$, $\|w_1\|_1 \leq 1.1$

$w_2 \in S$, $\|w_2\|_1 \leq 1.1$

$$w_\theta = \theta w_1 + (1-\theta) w_2$$

$w_\theta \in S$?

$\|w_\theta\|_1 \leq 1.1$

$$\|\theta w_1 + (1-\theta) w_2\|_1 \leq \theta \|w_1\|_1 + (1-\theta)\|w_2\|_1 \leq \theta \cdot 1.1 + (1-\theta) 1.1 \leq 1.1$$

Link to the code

Optimality conditions

Background

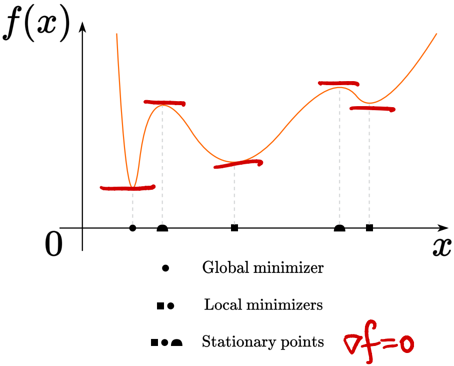
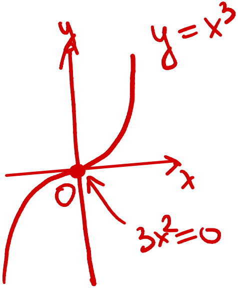


Figure 1: Illustration of different stationary (critical) points



$$f(x) \to \min_{x \in S}$$

MAX
MIN
седловая точка

Background

$$f(x) \to \min_{x \in S}$$

A set S is usually called a **budget set**.

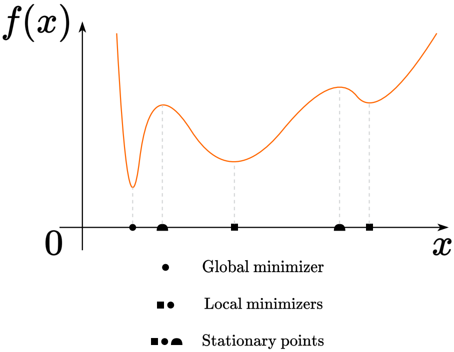


Figure 1: Illustration of different stationary (critical) points

Background

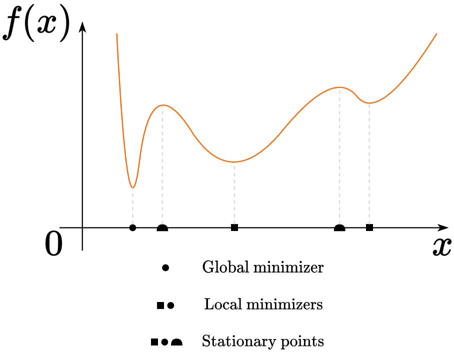


Figure 1: Illustration of different stationary (critical) points

$$f(x) \to \min_{x \in S}$$

A set S is usually called a **budget set**.
We say that the problem has a solution if the budget set **is not empty**: $x^* \in S$, in which the minimum or the infimum of the given function is achieved.

Background

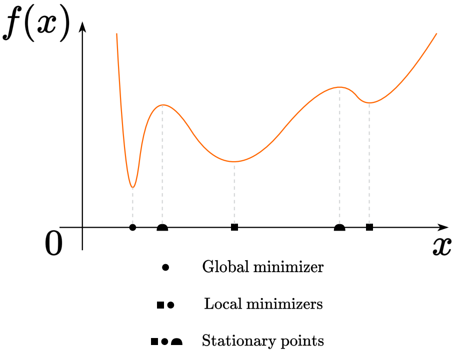


Figure 1: Illustration of different stationary (critical) points

$$f(x) \to \min_{x \in S}$$

A set S is usually called a **budget set**.
We say that the problem has a solution if the budget set **is not empty**: $x^* \in S$, in which the minimum or the infimum of the given function is achieved.

- A point x^* is a **global minimizer** if $f(x^*) \leq f(x)$ for all x.

Background

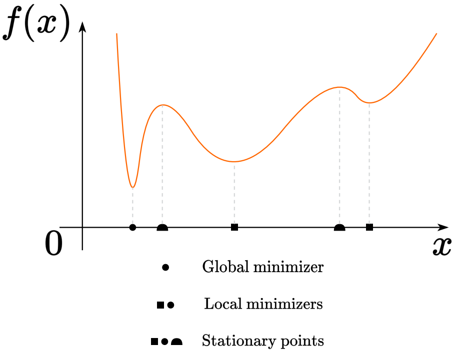


Figure 1: Illustration of different stationary (critical) points

$$f(x) \to \min_{x \in S}$$

A set S is usually called a **budget set**.

We say that the problem has a solution if the budget set **is not empty**: $x^* \in S$, in which the minimum or the infimum of the given function is achieved.

- A point x^* is a **global minimizer** if $f(x^*) \leq f(x)$ for all x.
- A point x^* is a **local minimizer** if there exists a neighborhood N of x^* such that $f(x^*) \leq f(x)$ for all $x \in N$.

Background

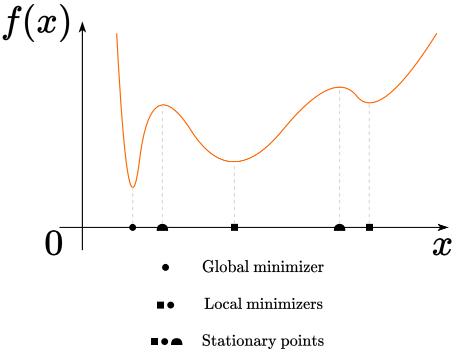


Figure 1: Illustration of different stationary (critical) points

$$f(x) \to \min_{x \in S}$$

A set S is usually called a **budget set**.
We say that the problem has a solution if the budget set **is not empty**: $x^* \in S$, in which the minimum or the infimum of the given function is achieved.

- A point x^* is a **global minimizer** if $f(x^*) \leq f(x)$ for all x.
- A point x^* is a **local minimizer** if there exists a neighborhood N of x^* such that $f(x^*) \leq f(x)$ for all $x \in N$.
- A point x^* is a **strict local minimizer** (also called a **strong local minimizer**) if there exists a neighborhood N of x^* such that $f(x^*) < f(x)$ for all $x \in N$ with $x \neq x^*$.

Background

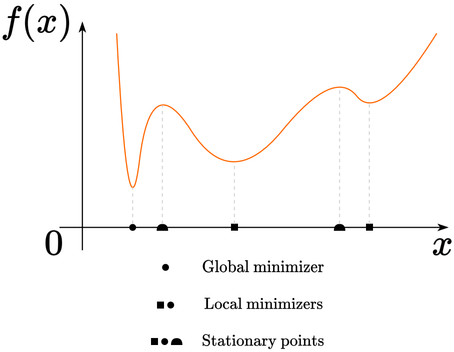


Figure 1: Illustration of different stationary (critical) points

$$f(x) \to \min_{x \in S}$$

A set S is usually called a **budget set**.

We say that the problem has a solution if the budget set **is not empty**: $x^* \in S$, in which the minimum or the infimum of the given function is achieved.

- A point x^* is a **global minimizer** if $f(x^*) \leq f(x)$ for all x.
- A point x^* is a **local minimizer** if there exists a neighborhood N of x^* such that $f(x^*) \leq f(x)$ for all $x \in N$.
- A point x^* is a **strict local minimizer** (also called a **strong local minimizer**) if there exists a neighborhood N of x^* such that $f(x^*) < f(x)$ for all $x \in N$ with $x \neq x^*$.
- We call x^* a **stationary point** (or critical) if $\nabla f(x^*) = 0$. Any local minimizer of a differentiable function must be a stationary point.

Extreme value (Weierstrass) theorem

Theorem

Let $S \subset \mathbb{R}^n$ be a compact set and $f(x)$ a continuous function on S. So, the point of the global minimum of the function $f(x)$ on S exists.

Extreme value (Weierstrass) theorem

Theorem

Let $S \subset \mathbb{R}^n$ be a compact set and $f(x)$ a continuous function on S. So, the point of the global minimum of the function $f(x)$ on S exists.

Figure 2: A lot of practical problems are theoretically solvable

Extreme value (Weierstrass) theorem

Theorem

Let $S \subset \mathbb{R}^n$ be a compact set and $f(x)$ a continuous function on S. So, the point of the global minimum of the function $f(x)$ on S exists.

Figure 2: A lot of practical problems are theoretically solvable

Taylor's Theorem

Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and that $p \in \mathbb{R}^n$. Then we have:

$$f(x+p) = f(x) + \nabla f(x+tp)^T p \quad \text{for some } t \in (0,1)$$

Extreme value (Weierstrass) theorem

Theorem

Let $S \subset \mathbb{R}^n$ be a compact set and $f(x)$ a continuous function on S. So, the point of the global minimum of the function $f(x)$ on S exists.

Figure 2: A lot of practical problems are theoretically solvable

Taylor's Theorem

Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and that $p \in \mathbb{R}^n$. Then we have:

$$f(x + p) = f(x) + \nabla f(x + tp)^T p \quad \text{for some } t \in (0, 1)$$

Moreover, if f is twice continuously differentiable, we have:

$$\nabla f(x + p) = \nabla f(x) + \int_0^1 \nabla^2 f(x + tp)p \, dt$$

$$f(x + p) = f(x) + \nabla f(x)^T p + \frac{1}{2} p^T \nabla^2 f(x + tp) p$$

for some $t \in (0, 1)$.

$$\min_{x \in \mathbb{R}^n} f(x)$$

Unconstrained optimization

Necessary Conditions

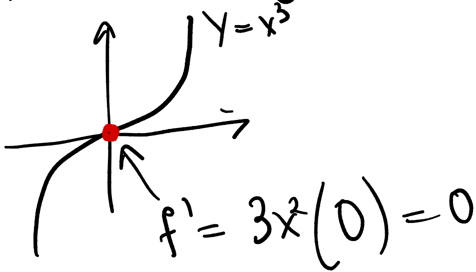
Если x^* - min, то $f'(x^*)=0$

First-Order Necessary Conditions

If x^* is a local minimizer and f is continuously differentiable in an open neighborhood, then

$$\nabla f(x^*) = 0$$

НЕОБХ. УСЛОВИЕ, НО НЕ ДОСТАТОЧНОЕ



Necessary Conditions

First-Order Necessary Conditions

If x^* is a local minimizer and f is continuously differentiable in an open neighborhood, then

$$\nabla f(x^*) = 0$$

Proof

Suppose for contradiction that $\nabla f(x^*) \neq 0$. Define the vector $p = -\nabla f(x^*)$ and note that

$$p^T \nabla f(x^*) = -\|\nabla f(x^*)\|^2 < 0$$

Necessary Conditions

First-Order Necessary Conditions

If x^* is a local minimizer and f is continuously differentiable in an open neighborhood, then

$$\nabla f(x^*) = 0$$

Proof
Suppose for contradiction that $\nabla f(x^*) \neq 0$. Define the vector $p = -\nabla f(x^*)$ and note that

$$p^T \nabla f(x^*) = -\|\nabla f(x^*)\|^2 < 0$$

Because ∇f is continuous near x^*, there is a scalar $T > 0$ such that

$$p^T \nabla f(x^* + tp) < 0, \text{ for all } t \in [0, T]$$

Necessary Conditions

First-Order Necessary Conditions

If x^* is a local minimizer and f is continuously differentiable in an open neighborhood, then

$$\nabla f(x^*) = 0$$

Proof

Suppose for contradiction that $\nabla f(x^*) \neq 0$. Define the vector $p = -\nabla f(x^*)$ and note that

$$p^T \nabla f(x^*) = -\|\nabla f(x^*)\|^2 < 0$$

Because ∇f is continuous near x^*, there is a scalar $T > 0$ such that

$$p^T \nabla f(x^* + tp) < 0, \text{ for all } t \in [0, T]$$

For any $\bar{t} \in (0, T]$, we have by Taylor's theorem that

$$f(x^* + \bar{t}p) = f(x^*) + \bar{t}p^T \nabla f(x^* + tp), \text{ for some } t \in (0, \bar{t})$$

Necessary Conditions

First-Order Necessary Conditions

If x^* is a local minimizer and f is continuously differentiable in an open neighborhood, then

$$\nabla f(x^*) = 0$$

Proof

Suppose for contradiction that $\nabla f(x^*) \neq 0$. Define the vector $p = -\nabla f(x^*)$ and note that

$$p^T \nabla f(x^*) = -\|\nabla f(x^*)\|^2 < 0$$

Because ∇f is continuous near x^*, there is a scalar $T > 0$ such that

$$p^T \nabla f(x^* + tp) < 0, \text{ for all } t \in [0, T]$$

For any $\bar{t} \in (0, T]$, we have by Taylor's theorem that

$$f(x^* + \bar{t}p) = f(x^*) + \bar{t}p^T \nabla f(x^* + tp), \text{ for some } t \in (0, \bar{t})$$

Therefore, $f(x^* + \bar{t}p) < f(x^*)$ for all $\bar{t} \in (0, T]$. We have found a direction from x^* along which f decreases, so x^* is not a local minimizer, leading to a contradiction.

Sufficient Conditions

достаточное условие

Second-Order Sufficient Conditions

Suppose that $\nabla^2 f$ is continuous in an open neighborhood of x^* and that

$$\boxed{\nabla f(x^*) = 0 \quad \nabla^2 f(x^*) \succ 0.}$$

Then x^* is a strict local minimizer of f.

Sufficient Conditions

Second-Order Sufficient Conditions

Suppose that $\nabla^2 f$ is continuous in an open neighborhood of x^* and that

$$\nabla f(x^*) = 0 \quad \nabla^2 f(x^*) \succ 0.$$

Then x^* is a strict local minimizer of f.

Proof

Because the Hessian is continuous and positive definite at x^*, we can choose a radius $r > 0$ such that $\nabla^2 f(x)$ remains positive definite for all x in the open ball $B = \{z \mid \|z - x^*\| < r\}$. Taking any nonzero vector p with $\|p\| < r$, we have $x^* + p \in B$ and so

Sufficient Conditions

Second-Order Sufficient Conditions

Suppose that $\nabla^2 f$ is continuous in an open neighborhood of x^* and that

$$\nabla f(x^*) = 0 \quad \nabla^2 f(x^*) \succ 0.$$

Then x^* is a strict local minimizer of f.

Proof

Because the Hessian is continuous and positive definite at x^*, we can choose a radius $r > 0$ such that $\nabla^2 f(x)$ remains positive definite for all x in the open ball $B = \{z \mid \|z - x^*\| < r\}$. Taking any nonzero vector p with $\|p\| < r$, we have $x^* + p \in B$ and so

$$f(x^* + p) = f(x^*) + p^T \nabla f(x^*) + \frac{1}{2} p^T \nabla^2 f(z) p$$

Sufficient Conditions

Second-Order Sufficient Conditions

Suppose that $\nabla^2 f$ is continuous in an open neighborhood of x^* and that

$$\nabla f(x^*) = 0 \quad \nabla^2 f(x^*) \succ 0.$$

Then x^* is a strict local minimizer of f.

Proof

Because the Hessian is continuous and positive definite at x^*, we can choose a radius $r > 0$ such that $\nabla^2 f(x)$ remains positive definite for all x in the open ball $B = \{z \mid \|z - x^*\| < r\}$. Taking any nonzero vector p with $\|p\| < r$, we have $x^* + p \in B$ and so

$$f(x^* + p) = f(x^*) + p^T \nabla f(x^*) + \frac{1}{2} p^T \nabla^2 f(z) p$$

$$= f(x^*) + \frac{1}{2} p^T \nabla^2 f(z) p$$

Sufficient Conditions

Second-Order Sufficient Conditions

Suppose that $\nabla^2 f$ is continuous in an open neighborhood of x^* and that

$$\nabla f(x^*) = 0 \quad \nabla^2 f(x^*) \succ 0.$$

Then x^* is a strict local minimizer of f.

Proof

Because the Hessian is continuous and positive definite at x^*, we can choose a radius $r > 0$ such that $\nabla^2 f(x)$ remains positive definite for all x in the open ball $B = \{z \mid \|z - x^*\| < r\}$. Taking any nonzero vector p with $\|p\| < r$, we have $x^* + p \in B$ and so

$$f(x^* + p) = f(x^*) + p^T \nabla f(x^*) + \frac{1}{2} p^T \nabla^2 f(z) p$$

$$= f(x^*) + \frac{1}{2} p^T \nabla^2 f(z) p$$

where $z = x^* + tp$ for some $t \in (0, 1)$. Since $z \in B$, we have $p^T \nabla^2 f(z) p > 0$, and therefore $f(x^* + p) > f(x^*)$, giving the result.

Peano counterexample

Note, that if $\nabla f(x^*) = 0, \nabla^2 f(x^*) \succeq 0$, i.e. the hessian is positive *semidefinite*, we cannot be sure if x^* is a local minimum.

Peano counterexample

Note, that if $\nabla f(x^*) = 0, \nabla^2 f(x^*) \succeq 0$, i.e. the hessian is positive *semidefinite*, we cannot be sure if x^* is a local minimum.

$$f(x, y) = (2x^2 - y)(x^2 - y)$$

Peano counterexample

Note, that if $\nabla f(x^*) = 0, \nabla^2 f(x^*) \succeq 0$, i.e. the hessian is positive *semidefinite*, we cannot be sure if x^* is a local minimum.

$$f(x, y) = (2x^2 - y)(x^2 - y)$$

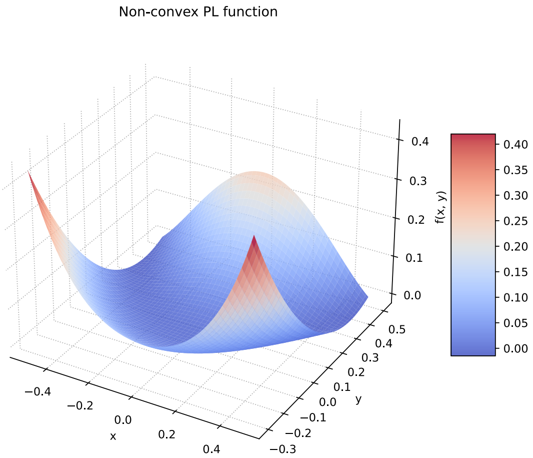
Although the surface does not have a local minimizer at the origin, its intersection with any vertical plane through the origin (a plane with equation $y = mx$ or $x = 0$) is a curve that has a local minimum at the origin. In other words, if a point starts at the origin $(0, 0)$ of the plane, and moves away from the origin along any straight line, the value of $(2x^2 - y)(x^2 - y)$ will increase at the start of the motion. Nevertheless, $(0, 0)$ is not a local minimizer of the function, because moving along a parabola such as $y = \sqrt{2}x^2$ will cause the function value to decrease.

Peano counterexample

Note, that if $\nabla f(x^*) = 0, \nabla^2 f(x^*) \succeq 0$, i.e. the hessian is positive *semidefinite*, we cannot be sure if x^* is a local minimum.

$$f(x, y) = (2x^2 - y)(x^2 - y)$$

Although the surface does not have a local minimizer at the origin, its intersection with any vertical plane through the origin (a plane with equation $y = mx$ or $x = 0$) is a curve that has a local minimum at the origin. In other words, if a point starts at the origin $(0, 0)$ of the plane, and moves away from the origin along any straight line, the value of $(2x^2 - y)(x^2 - y)$ will increase at the start of the motion. Nevertheless, $(0, 0)$ is not a local minimizer of the function, because moving along a parabola such as $y = \sqrt{2}x^2$ will cause the function value to decrease.



Constrained optimization

General first-order local optimality condition

Direction $d \in \mathbb{R}^n$ is a feasible direction at $x^* \in S \subseteq \mathbb{R}^n$ if small steps along d do not take us outside of S.

General first-order local optimality condition

Direction $d \in \mathbb{R}^n$ is a feasible direction at $x^* \in S \subseteq \mathbb{R}^n$ if small steps along d do not take us outside of S.

Consider a set $S \subseteq \mathbb{R}^n$ and a function $f : \mathbb{R}^n \to \mathbb{R}$. Suppose that $x^* \in S$ is a point of local minimum for f over S, and further assume that f is continuously differentiable around x^*.

General first-order local optimality condition

Direction $d \in \mathbb{R}^n$ is a feasible direction at $x^* \in S \subseteq \mathbb{R}^n$ if small steps along d do not take us outside of S.

Consider a set $S \subseteq \mathbb{R}^n$ and a function $f : \mathbb{R}^n \to \mathbb{R}$. Suppose that $x^* \in S$ is a point of local minimum for f over S, and further assume that f is continuously differentiable around x^*.

1. Then for every feasible direction $d \in \mathbb{R}^n$ at x^* it holds that $\nabla f(x^*)^\top d \geq 0$.

General first-order local optimality condition

Direction $d \in \mathbb{R}^n$ is a feasible direction at $x^* \in S \subseteq \mathbb{R}^n$ if small steps along d do not take us outside of S.

Consider a set $S \subseteq \mathbb{R}^n$ and a function $f : \mathbb{R}^n \to \mathbb{R}$. Suppose that $x^* \in S$ is a point of local minimum for f over S, and further assume that f is continuously differentiable around x^*.

1. Then for every feasible direction $d \in \mathbb{R}^n$ at x^* it holds that $\nabla f(x^*)^\top d \geq 0$.
2. If, additionally, S is convex then

$$
\nabla f(x^*)^\top (x - x^*) \geq 0, \forall x \in S.
$$

General first-order local optimality condition

Direction $d \in \mathbb{R}^n$ is a feasible direction at $x^* \in S \subseteq \mathbb{R}^n$ if small steps along d do not take us outside of S.

Consider a set $S \subseteq \mathbb{R}^n$ and a function $f : \mathbb{R}^n \to \mathbb{R}$. Suppose that $x^* \in S$ is a point of local minimum for f over S, and further assume that f is continuously differentiable around x^*.

1. Then for every feasible direction $d \in \mathbb{R}^n$ at x^* it holds that $\nabla f(x^*)^\top d \geq 0$.
2. If, additionally, S is convex then

$$\nabla f(x^*)^\top (x - x^*) \geq 0, \forall x \in S.$$

General first-order local optimality condition

Direction $d \in \mathbb{R}^n$ is a feasible direction at $x^* \in S \subseteq \mathbb{R}^n$ if small steps along d do not take us outside of S.

Consider a set $S \subseteq \mathbb{R}^n$ and a function $f : \mathbb{R}^n \to \mathbb{R}$. Suppose that $x^* \in S$ is a point of local minimum for f over S, and further assume that f is continuously differentiable around x^*.

1. Then for every feasible direction $d \in \mathbb{R}^n$ at x^* it holds that $\nabla f(x^*)^\top d \geq 0$.
2. If, additionally, S is convex then

$$\nabla f(x^*)^\top (x - x^*) \geq 0, \forall x \in S.$$

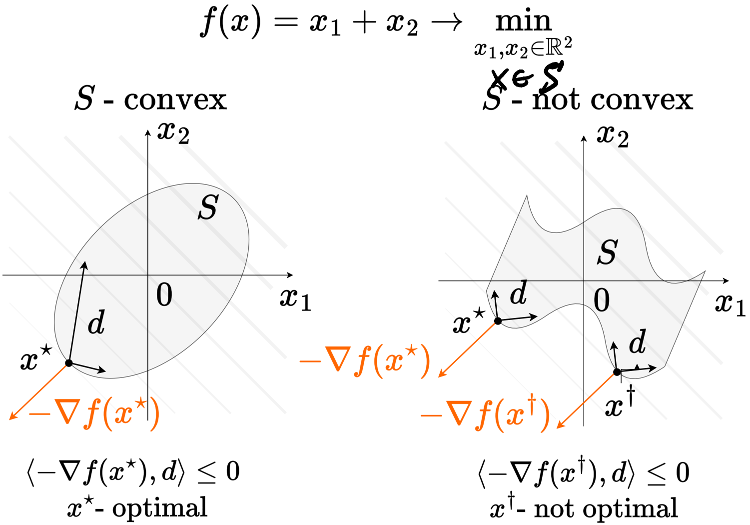
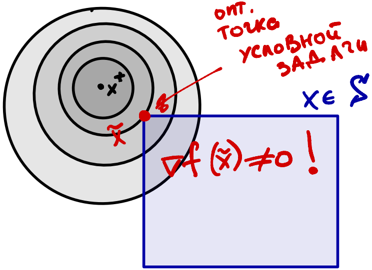


Figure 3: General first order local optimality condition

Convex case

It should be mentioned, that in the **convex** case (i.e., $f(x)$ is convex) necessary condition becomes sufficient.



Convex case

It should be mentioned, that in the **convex** case (i.e., $f(x)$ is convex) necessary condition becomes sufficient.

One more important result for the convex unconstrained case sounds as follows. If $f(x) : S \to \mathbb{R}$ - convex function defined on the convex set S, then:

Convex case

It should be mentioned, that in the **convex** case (i.e., $f(x)$ is convex) necessary condition becomes sufficient.

One more important result for the convex unconstrained case sounds as follows. If $f(x) : S \to \mathbb{R}$ - convex function defined on the convex set S, then:

- Any local minima is the global one.

Convex case

It should be mentioned, that in the **convex** case (i.e., $f(x)$ is convex) necessary condition becomes sufficient.

One more important result for the convex unconstrained case sounds as follows. If $f(x) : S \to \mathbb{R}$ - convex function defined on the convex set S, then:

- Any local minima is the global one.
- The set of the local minimizers S^* is convex.

Convex case

It should be mentioned, that in the **convex** case (i.e., $f(x)$ is convex) necessary condition becomes sufficient.

One more important result for the convex unconstrained case sounds as follows. If $f(x) : S \to \mathbb{R}$ - convex function defined on the convex set S, then:

- Any local minima is the global one.
- The set of the local minimizers S^* is convex.
- If $f(x)$ - strictly or strongly convex function, then S^* contains only one single point $S^* = \{x^*\}$.

Optimization with equality constraints

Things are pretty simple and intuitive in unconstrained problems. In this section, we will add one equality constraint, i.e.

Optimization with equality constraints

Things are pretty simple and intuitive in unconstrained problems. In this section, we will add one equality constraint, i.e.

$$
\begin{split}
& f(x) \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } & h(x) = 0
\end{split}
$$

Optimization with equality constraints

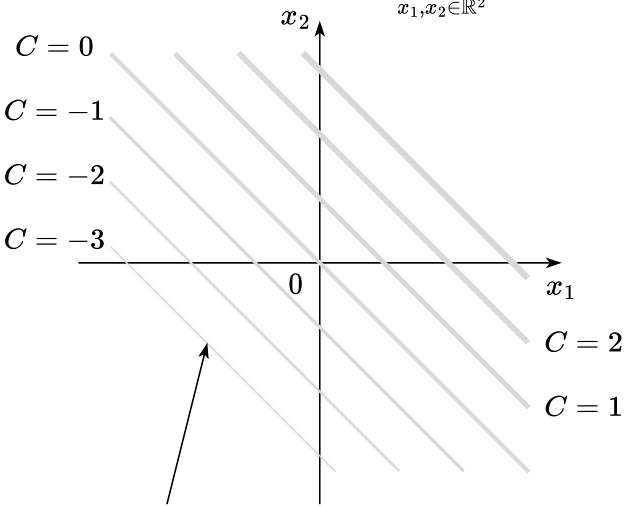
Things are pretty simple and intuitive in unconstrained problems. In this section, we will add one equality constraint, i.e.

$$
\begin{split}
& f(x) \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } & h(x) = 0
\end{split}
$$

We will try to illustrate an approach to solve this problem through the simple example with $f(x) = x_1 + x_2$ and $h(x) = x_1^2 + x_2^2 - 2$.

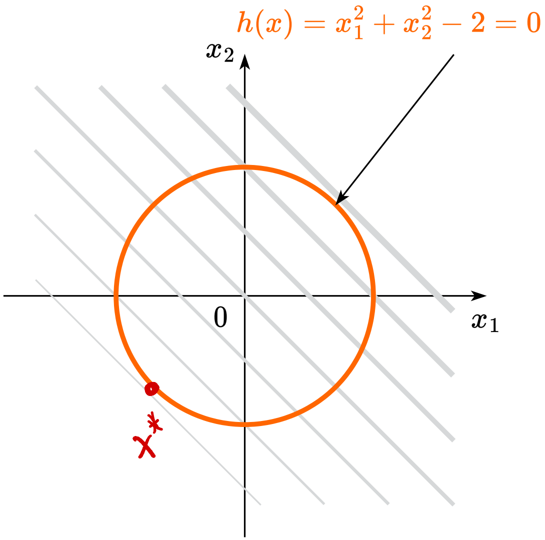
Optimization with equality constraints

$$f(x) = x_1 + x_2 \to \min_{x_1, x_2 \in \mathbb{R}^2}$$

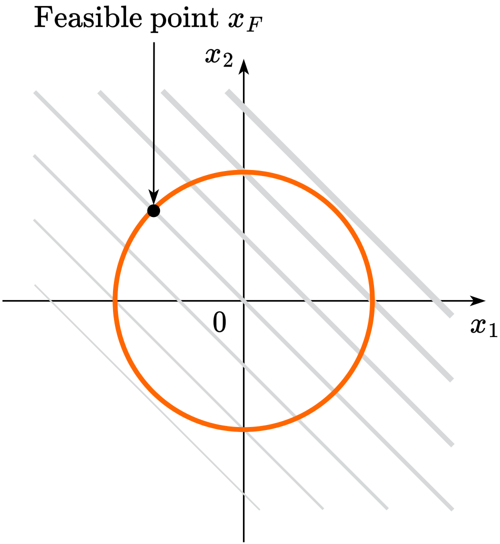


Contour lines of $f(x) = x_1 + x_2 = C$

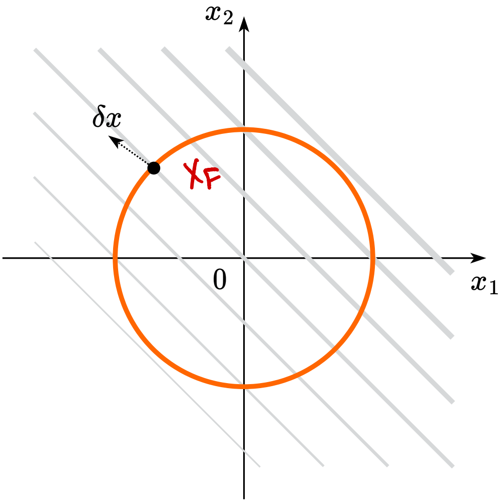
Optimization with equality constraints



Optimization with equality constraints

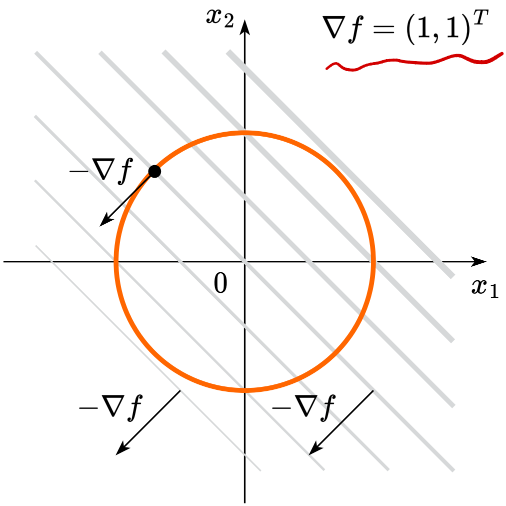


Optimization with equality constraints



Optimization with equality constraints

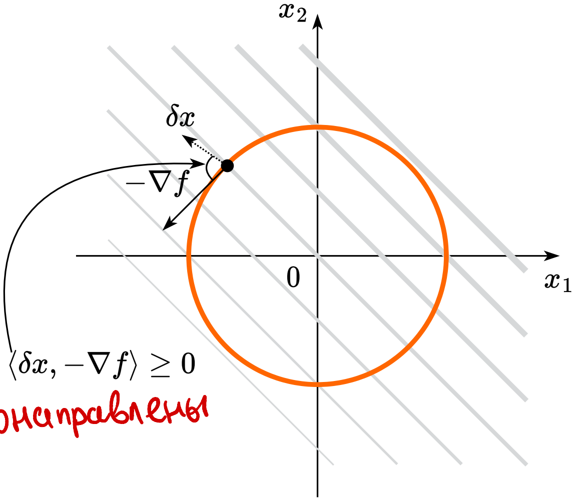
$f(x_1, x_2) = x_1 + x_2$



Optimization with equality constraints

накладываем на δx требование $f \downarrow\downarrow$

We want: $f(x_F + \delta x) \leq f(x_F)$



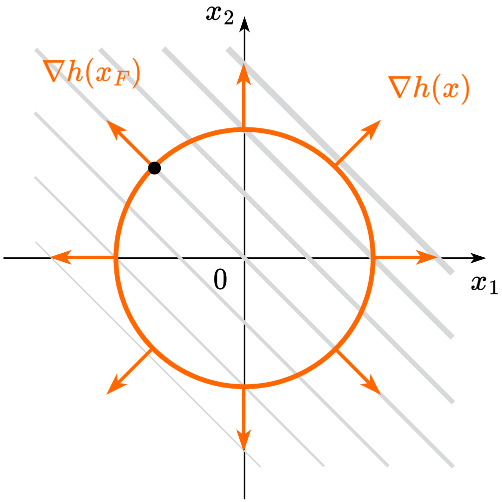
δx, $-\nabla f$ сонаправлены

Optimization with equality constraints

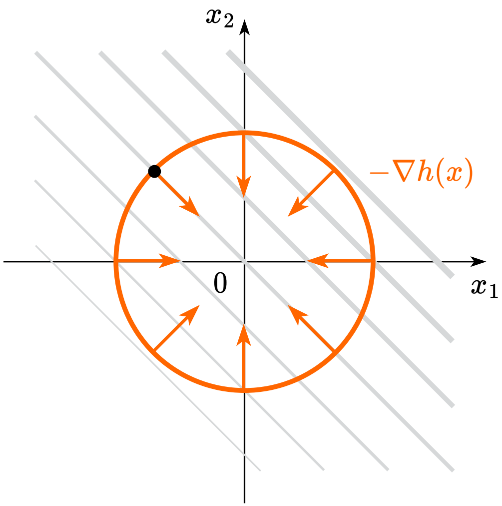
$h(x) = x_1^2 + x_2^2 - 2$

$h(x) = 0$

$$\nabla h = (2x_1, 2x_2)^T$$

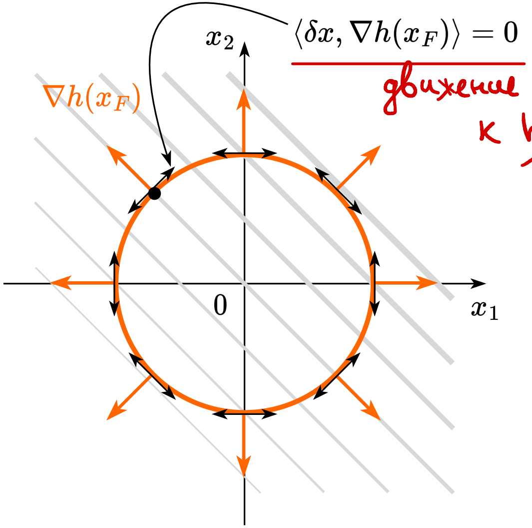


Optimization with equality constraints



Optimization with equality constraints

ортогональны нормали



движение по касательной к $h(x) = 0$

Optimization with equality constraints

Generally: to move from x_F along the budget set toward decreasing the function, we need to guarantee two conditions:

Optimization with equality constraints

Generally: to move from x_F along the budget set toward decreasing the function, we need to guarantee two conditions:

$$\langle \delta x, \nabla h(x_F) \rangle = 0$$

Optimization with equality constraints

Generally: to move from x_F along the budget set toward decreasing the function, we need to guarantee two conditions:

$$\langle \delta x, \nabla h(x_F) \rangle = 0$$

$$\langle \delta x, -\nabla f(x_F) \rangle > 0$$

Optimization with equality constraints

Generally: to move from x_F along the budget set toward decreasing the function, we need to guarantee two conditions:

$$\langle \delta x, \nabla h(x_F) \rangle = 0$$

$$\langle \delta x, -\nabla f(x_F) \rangle > 0$$

Let's assume, that in the process of such a movement, we have come to the point where

Optimization with equality constraints

Generally: to move from x_F along the budget set toward decreasing the function, we need to guarantee two conditions:

$$\langle \delta x, \nabla h(x_F) \rangle = 0$$

$$\langle \delta x, -\nabla f(x_F) \rangle > 0$$

Let's assume, that in the process of such a movement, we have come to the point where

$$-\nabla f(x) = \nu \nabla h(x)$$

Optimization with equality constraints

Generally: to move from x_F along the budget set toward decreasing the function, we need to guarantee two conditions:

$$\langle \delta x, \nabla h(x_F) \rangle = 0$$

$$\langle \delta x, -\nabla f(x_F) \rangle > 0$$

Let's assume, that in the process of such a movement, we have come to the point where

$$-\nabla f(x) = \nu \nabla h(x)$$

$$\langle \delta x, -\nabla f(x) \rangle = \langle \delta x, \nu \nabla h(x) \rangle = 0$$

Optimization with equality constraints

Generally: to move from x_F along the budget set toward decreasing the function, we need to guarantee two conditions:

$$\langle \delta x, \nabla h(x_F) \rangle = 0$$

$$\langle \delta x, -\nabla f(x_F) \rangle > 0$$

СТАНЕТ $\langle \delta x, -\nabla f(x_F) \rangle =$

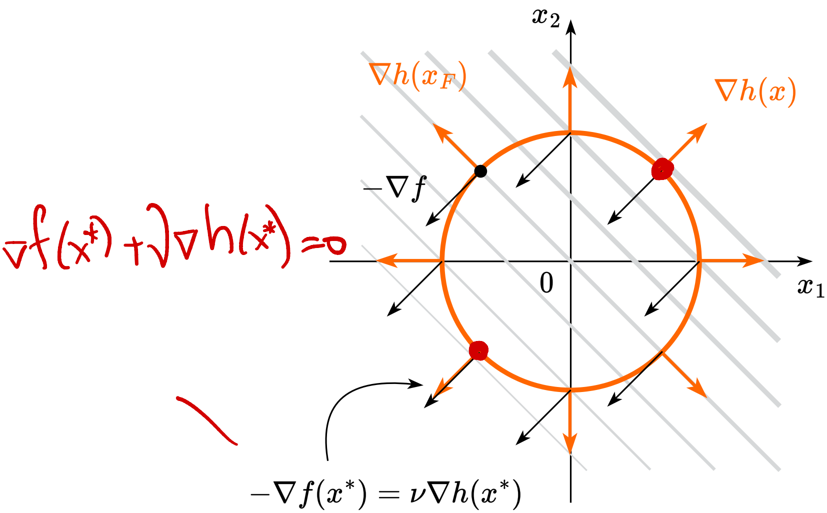
Let's assume, that in the process of such a movement, we have come to the point where

$$-\nabla f(x) = \nu \nabla h(x)$$

$$\langle \delta x, -\nabla f(x) \rangle = \langle \delta x, \nu \nabla h(x) \rangle = 0$$

Then we came to the point of the budget set, moving from which it will not be possible to reduce our function. This is the local minimum in the constrained problem :)

Optimization with equality constraints



Lagrangian

So let's define a Lagrange function (just for our convenience):

$$L(x, \nu) = f(x) + \nu h(x)$$

Lagrangian

$f : \mathbb{R}^n \to \mathbb{R}$

So let's define a Lagrange function (just for our convenience):

$L : \mathbb{R}^{n+1} \to \mathbb{R}$

$$L(x, \nu) = f(x) + \nu h(x)$$

Then if the problem is *regular* (we will define it later) and the point x^* is the local minimum of the problem described above, then there exists ν^*:

Lagrangian

So let's define a Lagrange function (just for our convenience):

$$L(x, \nu) = f(x) + \nu h(x)$$

Then if the problem is *regular* (we will define it later) and the point x^* is the local minimum of the problem described above, then there exists ν^*:

Necessary conditions

We should notice that $L(x^*, \nu^*) = f(x^*)$.

Lagrangian

So let's define a Lagrange function (just for our convenience):

$$L(x, \nu) = f(x) + \nu h(x)$$

Then if the problem is *regular* (we will define it later) and the point x^* is the local minimum of the problem described above, then there exists ν^*:

Necessary conditions

$\nabla_x L(x^*, \nu^*) = 0$ that's written above

We should notice that $L(x^*, \nu^*) = f(x^*)$.

Lagrangian

$$\nabla_x L = \nabla f(x) + \nu \nabla h(x)$$

$$\nabla_\nu L = h(x)$$

So let's define a Lagrange function (just for our convenience):

$$L(x, \nu) = f(x) + \nu h(x)$$

Then if the problem is *regular* (we will define it later) and the point x^* is the local minimum of the problem described above, then there exists ν^*:

Necessary conditions

$\nabla_x L(x^*, \nu^*) = 0$ that's written above оптимальность

$\nabla_\nu L(x^*, \nu^*) = 0$ budget constraint $h(x) = 0$

We should notice that $L(x^*, \nu^*) = f(x^*)$.

Lagrangian

So let's define a Lagrange function (just for our convenience):

$$\min_{\substack{x \in \mathbb{R}^n \\ \nu \in \mathbb{R}}} L(x, \nu)$$

$$L(x, \nu) = f(x) + \nu h(x)$$

Then if the problem is *regular* (we will define it later) and the point x^* is the local minimum of the problem described above, then there exists ν^*:

Necessary conditions

$\nabla_x L(x^*, \nu^*) = 0$ that's written above

$\nabla_\nu L(x^*, \nu^*) = 0$ budget constraint

Sufficient conditions

We should notice that $L(x^*, \nu^*) = f(x^*)$.

Lagrangian

So let's define a Lagrange function (just for our convenience):

$$L(x, \nu) = f(x) + \nu h(x)$$

Then if the problem is *regular* (we will define it later) and the point x^* is the local minimum of the problem described above, then there exists ν^*:

Necessary conditions

$\nabla_x L(x^*, \nu^*) = 0$ that's written above

$\nabla_\nu L(x^*, \nu^*) = 0$ budget constraint

Sufficient conditions

$\langle y, \nabla^2_{xx} L(x^*, \nu^*) y \rangle > 0,$

We should notice that $L(x^*, \nu^*) = f(x^*)$.

Lagrangian

So let's define a Lagrange function (just for our convenience):

$$L(x, \nu) = f(x) + \nu h(x)$$

Then if the problem is *regular* (we will define it later) and the point x^* is the local minimum of the problem described above, then there exists ν^*:

Necessary conditions

$\nabla_x L(x^*, \nu^*) = 0$ that's written above

$\nabla_\nu L(x^*, \nu^*) = 0$ budget constraint

Sufficient conditions

$\langle y, \nabla^2_{xx} L(x^*, \nu^*) y \rangle > 0,$

$\forall y \neq 0 \in \mathbb{R}^n : \nabla h(x^*)^\top y = 0$

ЕСЛИ ЗАДАЧА ВЫПУКЛАЯ, ТО НЕОБХ. СТАНОВЯТСЯ ДОСТАТОЧНЫМИ

We should notice that $L(x^*, \nu^*) = f(x^*)$.

Equality constrained problem

$$
\begin{aligned}
& f(x) \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } & h_i(x) = 0, \; i = 1, \ldots, p
\end{aligned}
\tag{ECP}
$$

$$
L(x, \nu) = f(x) + \sum_{i=1}^{p} \nu_i h_i(x) = f(x) + \nu^\top h(x)
$$

Let $f(x)$ and $h_i(x)$ be twice differentiable at the point x^* and continuously differentiable in some neighborhood x^*. The local minimum conditions for $x \in \mathbb{R}^n, \nu \in \mathbb{R}^p$ are written as

ECP: Necessary conditions

$$
\begin{aligned}
\nabla_x L(x^*, \nu^*) = 0 \\
\nabla_\nu L(x^*, \nu^*) = 0
\end{aligned}
$$

ECP: Sufficient conditions

$$
\langle y, \nabla^2_{xx} L(x^*, \nu^*) y \rangle > 0,
$$

$$
\forall y \neq 0 \in \mathbb{R}^n : \nabla h_i(x^*)^\top y = 0
$$

Linear Least Squares

$Ax = b$ $\qquad \|x\|_2^2 \to \min\limits_{x \in \mathbb{R}^n}$, $Ax = b$

задача выпуклая!

a_1^T, a_2^T, a_3^T

$h(x) = Ax - b$ $\qquad h(x) = 0$

$a_1^T x = b_1$, $a_2^T x = b_2$, $a_3^T x = b_3$

Example

Pose the optimization problem and solve them for linear system $Ax = b$, $A \in \mathbb{R}^{m \times n}$ for three cases (assuming the matrix is full rank):

- $m < n$

m, n — недоопр. система. Решений бесконечно много

Решение: 1) $L(x, \nu) = x^T x + \nu^T (Ax - b) = \sum\limits_{i=1}^{n} x_i^2 + \sum\limits_{i=1}^{m} \nu_i (a_i^T x - b_i)$

2) Считаем $\nabla_x L(x, \nu)$ $\quad dL = d\left(x^T x + \nu^T (Ax - b)\right) = d\left(\langle x, x \rangle + \langle \nu, Ax - b \rangle\right) =$

$= \langle 2x, dx \rangle + \langle \nu, A dx \rangle = \langle 2x + A^T \nu, dx \rangle \Rightarrow \nabla_x L = A^T \nu + 2x$

$\nabla_x L = 0$

$A^T \nu + 2x = 0$

Linear Least Squares

2) $\nabla_\nu L = Ax - b = 0$

$A^T\nu + 2x = 0$

$$\begin{cases} A^T\nu + 2x = 0 \\ Ax - b = 0 \end{cases} \rightarrow \begin{cases} x = -\frac{1}{2}A^T\nu \\ A\cdot\left(-\frac{1}{2}A^T\nu\right) - b = 0 \end{cases}$$

$m < n$

$$\begin{cases} x = -\frac{1}{2}A^T\nu \\ -\frac{1}{2}\underbrace{AA^T}_{m\times n\; n\times m}\nu = b \end{cases} \rightarrow \boxed{\nu = -2(AA^T)^{-1}b}$$

$x = -\frac{1}{2}A^T(-2)(AA^T)^{-1}b$

$$\boxed{x = A^T(AA^T)^{-1}\cdot b}$$

Example

Pose the optimization problem and solve them for linear system $Ax = b$, $A \in \mathbb{R}^{m\times n}$ for three cases (assuming the matrix is full rank):

- $m < n$

$$A^\dagger = \lim_{\alpha\to 0} A^T(AA^T + \alpha\cdot I)^{-1}$$

Linear Least Squares

$$Ax = b \qquad A^{+} = A^{-1}$$

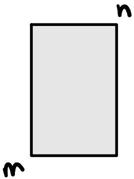
$$x^* = A^{-1} b$$

Example

Pose the optimization problem and solve them for linear system $Ax = b, A \in \mathbb{R}^{m \times n}$ for three cases (assuming the matrix is full rank):

- $m < n$
- $m = n$

Linear Least Squares



$x = 2$

$x = 3$

Example

Pose the optimization problem and solve them for linear system $Ax = b$, $A \in \mathbb{R}^{m \times n}$ for three cases (assuming the matrix is full rank):

- $m < n$
- $m = n$
- $m > n$

$$\|Ax - b\|_2^2 \to \min_{x \in \mathbb{R}^n}$$

\dagger

$$\nabla f = 2A^T(Ax - b) = 0$$

$$A^T A x = A^T b$$

$$\boxed{x = (A^T A)^{-1} A^T b} = A^\dagger b$$

$$\boxed{x = A^\dagger b}$$

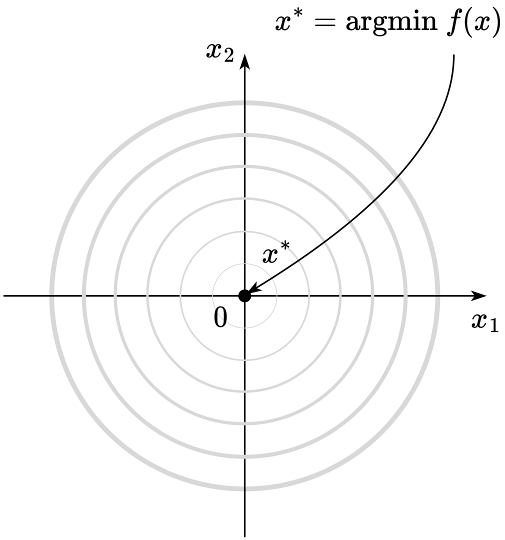
Optimization with inequality constraints

Example of inequality constraints

$$f(x) = x_1^2 + x_2^2 \quad g(x) = x_1^2 + x_2^2 - 1$$

$$f(x) \to \min_{x \in \mathbb{R}^n}$$
$$\text{s.t. } g(x) \le 0$$

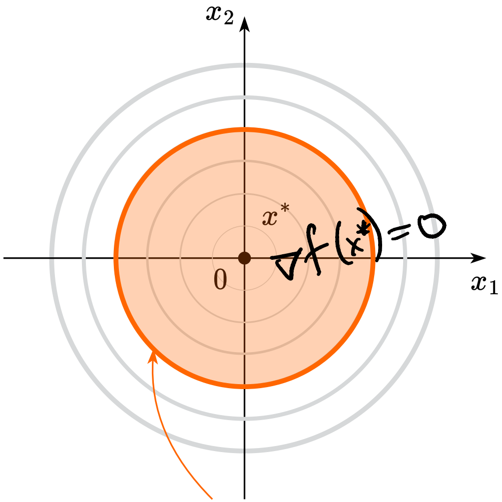
Optimization with inequality constraints



Contour lines of $f(x) = x_1^2 + x_2^2 = C$

Optimization with inequality constraints

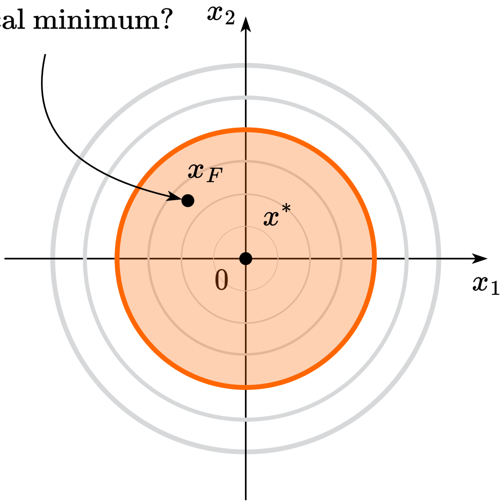
оптимум безусловной задачи совпадает с оптимумом условной



Feasible region $g(x) = x_1^2 + x_2^2 - 1 \leq 0$

Optimization with inequality constraints

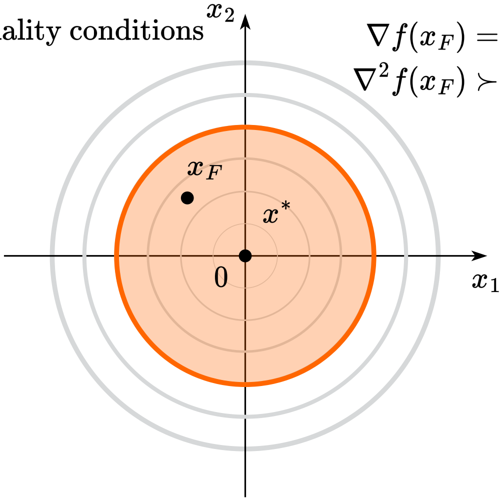
How to recognize that some feasible point is at local minimum?



Optimization with inequality constraints

Easy in this case! Just check unconstrained optimality conditions

$$\nabla f(x_F) = 0$$
$$\nabla^2 f(x_F) \succ 0$$



Optimization with inequality constraints

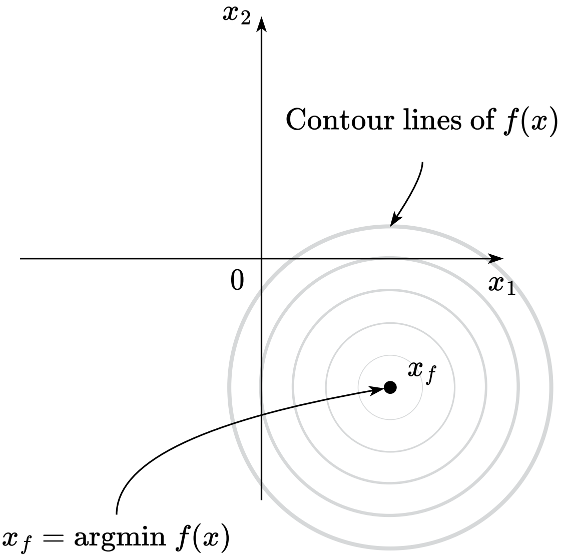
Thus, if the constraints of the type of inequalities are inactive in the constrained problem, then don't worry and write out the solution to the unconstrained problem. However, this is not the whole story. Consider the second childish example

$$f(x) = (x_1 - 1)^2 + (x_2 + 1)^2 \quad g(x) = x_1^2 + x_2^2 - 1$$

$$\begin{split} & f(x) \to \min_{x \in \mathbb{R}^n} \\ \text{s.t. } & g(x) \leq 0 \end{split}$$

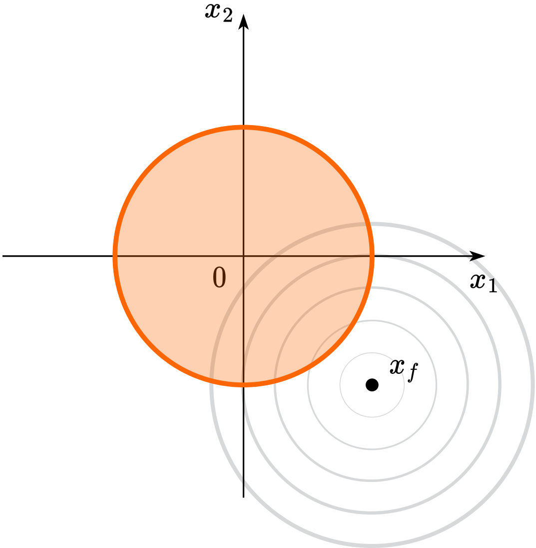
Optimization with inequality constraints

$$f(x) = (x_1 - 1)^2 + (x_2 + 1)^2 = C$$



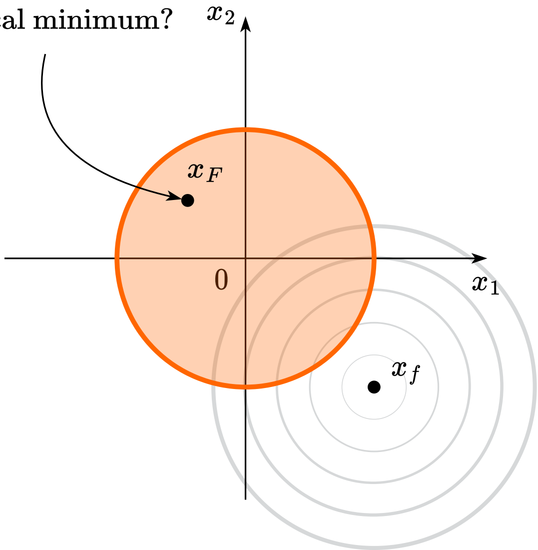
Optimization with inequality constraints

Feasible region $g(x) = x_1^2 + x_2^2 - 1 \leq 0$



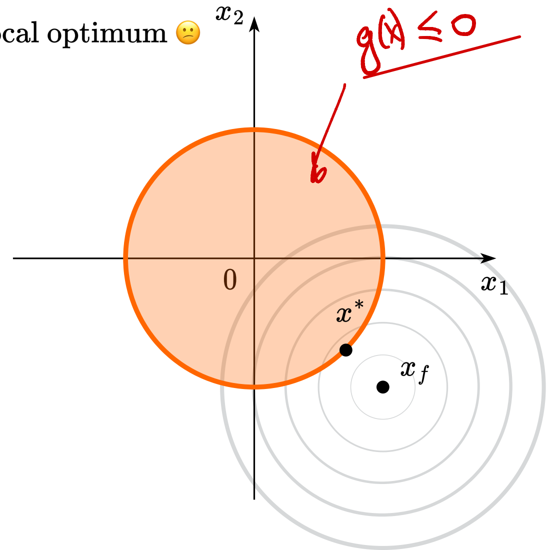
Optimization with inequality constraints

How to recognize that some feasible point is at local minimum?



Optimization with inequality constraints

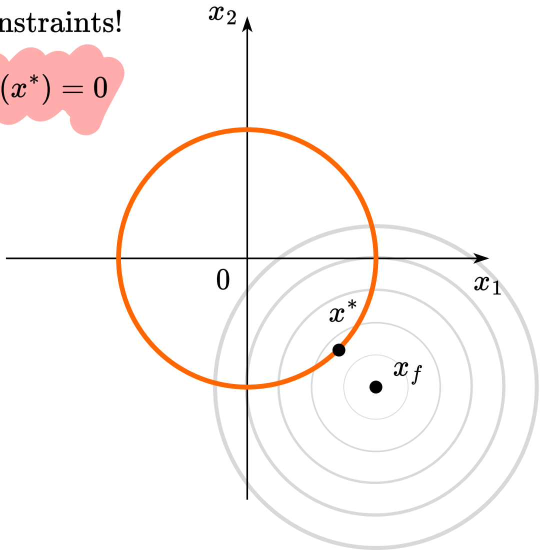
Not very easy in this case! Even gradient $\neq 0$ at local optimum 😕



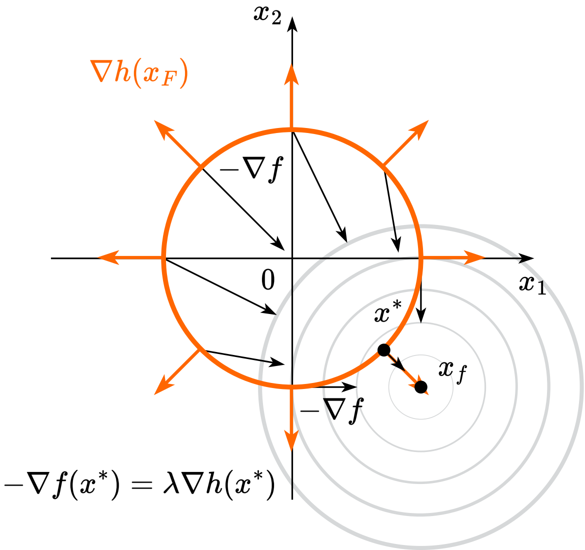
Optimization with inequality constraints

Effectively have a problem with equality constraints!

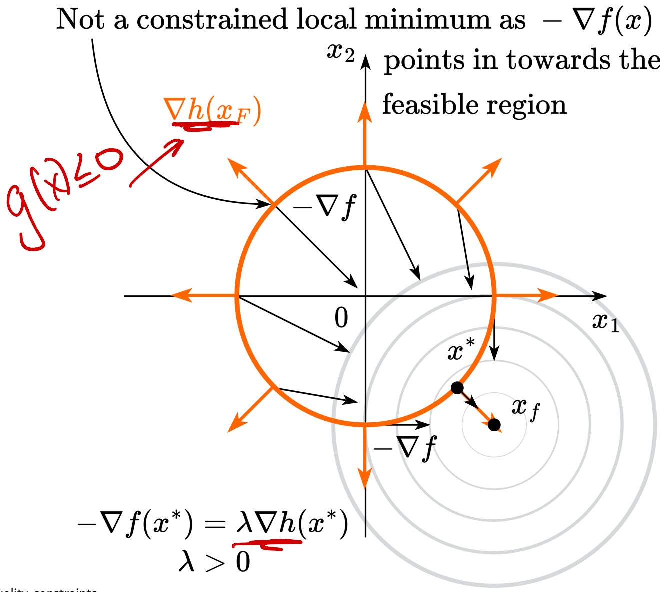
$g(x^*) = 0$



Optimization with inequality constraints



Optimization with inequality constraints



$$-\nabla f(x^*) = \lambda \nabla h(x^*)$$
$$\lambda > 0$$

Optimization with inequality constraints

So, we have a problem:

$$
f(x) \to \min_{x \in \mathbb{R}^n}
$$
$$
\text{s.t. } g(x) \leq 0
$$

Two possible cases:

$g(x) \leq 0$ is inactive. $g(x^*) < 0$

- $g(x^*) < 0$

Optimization with inequality constraints

So, we have a problem:

ограничение

не активно

$$
f(x) \to \min_{x \in \mathbb{R}^n}
$$

$$
\text{s.t. } g(x) \leq 0
$$

Two possible cases:

$g(x) \leq 0$ is inactive. $g(x^*) < 0$

- $g(x^*) < 0$
- $\nabla f(x^*) = 0$

Optimization with inequality constraints

So, we have a problem:

$$f(x) \to \min_{x \in \mathbb{R}^n}$$
$$\text{s.t. } g(x) \leq 0$$

x* лежит строго внутри S

Two possible cases:

$g(x) \leq 0$ is inactive. $g(x^*) < 0$

- $g(x^*) < 0$
- $\nabla f(x^*) = 0$
- $\nabla^2 f(x^*) > 0$

Optimization with inequality constraints

So, we have a problem:

$$
f(x) \to \min_{x \in \mathbb{R}^n}
$$

$$
\text{s.t. } g(x) \leq 0
$$

Two possible cases:

$g(x) \leq 0$ is inactive. $g(x^*) < 0$

- $g(x^*) < 0$
- $\nabla f(x^*) = 0$
- $\nabla^2 f(x^*) > 0$

Optimization with inequality constraints

So, we have a problem:

$$
f(x) \to \min_{x \in \mathbb{R}^n}
$$

$$
\text{s.t. } g(x) \leq 0
$$

ограничение активно

Two possible cases:

$g(x) \leq 0$ is inactive. $g(x^*) < 0$

- $g(x^*) < 0$
- $\nabla f(x^*) = 0$
- $\nabla^2 f(x^*) > 0$

$g(x) \leq 0$ is active. $g(x^*) = 0$

- $g(x^*) = 0$

x* лежит на границе S

Optimization with inequality constraints

So, we have a problem:

$$
f(x) \to \min_{x \in \mathbb{R}^n}
$$
$$
\text{s.t. } g(x) \leq 0
$$

Two possible cases:

$g(x) \leq 0$ is inactive. $g(x^*) < 0$

- $g(x^*) < 0$
- $\nabla f(x^*) = 0$
- $\nabla^2 f(x^*) > 0$

$g(x) \leq 0$ is active. $g(x^*) = 0$

- $g(x^*) = 0$
- Necessary conditions: $-\nabla f(x^*) = \lambda \nabla g(x^*)$, $\lambda > 0$

Optimization with inequality constraints

So, we have a problem:

$$
f(x) \to \min_{x \in \mathbb{R}^n}
$$

$$
\text{s.t. } g(x) \leq 0
$$

Two possible cases:

$g(x) \leq 0$ is inactive. $g(x^*) < 0$

- $g(x^*) < 0$
- $\nabla f(x^*) = 0$
- $\nabla^2 f(x^*) > 0$

$g(x) \leq 0$ is active. $g(x^*) = 0$

- $g(x^*) = 0$
- Necessary conditions: $-\nabla f(x^*) = \lambda \nabla g(x^*)$, $\lambda > 0$
- Sufficient conditions: $\langle y, \nabla^2_{xx} L(x^*, \lambda^*) y \rangle > 0, \forall y \neq 0 \in \mathbb{R}^n : \nabla g(x^*)^\top y = 0$

Lagrange function for inequality constraints

Combining two possible cases, we can write down the general conditions for the problem:

$$
\begin{aligned}
&f(x) \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } &g(x) \leq 0
\end{aligned}
$$

Let's define the Lagrange function:

$$
L(x, \lambda) = f(x) + \lambda g(x)
$$

The classical Karush-Kuhn-Tucker first and second-order optimality conditions for a local minimizer x^*, stated under some regularity conditions, can be written as follows.

Lagrange function for inequality constraints

Combining two possible cases, we can write down the general conditions for the problem:

$$
\begin{split}
& f(x) \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } & g(x) \leq 0
\end{split}
$$

Let's define the Lagrange function:

$$
L(x, \lambda) = f(x) + \lambda g(x)
$$

The classical Karush-Kuhn-Tucker first and second-order optimality conditions for a local minimizer x^*, stated under some regularity conditions, can be written as follows.

If x^* is a local minimum of the problem described above, then there exists a unique Lagrange multiplier λ^* such that:

$$
(1)\ \nabla_x L(x^*, \lambda^*) = 0
$$

Lagrange function for inequality constraints

Combining two possible cases, we can write down the general conditions for the problem:

$$
\begin{split}
& f(x) \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } & g(x) \leq 0
\end{split}
$$

Let's define the Lagrange function:

$$
L(x, \lambda) = f(x) + \lambda g(x)
$$

The classical Karush-Kuhn-Tucker first and second-order optimality conditions for a local minimizer x^*, stated under some regularity conditions, can be written as follows.

If x^* is a local minimum of the problem described above, then there exists a unique Lagrange multiplier λ^* such that:

$$
\begin{split}
& (1) \; \nabla_x L (x^*, \lambda^*) = 0 \\
& (2) \; \lambda^* \geq 0
\end{split}
$$

Lagrange function for inequality constraints

Combining two possible cases, we can write down the general conditions for the problem:

$$
\begin{split}
& f(x) \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } & g(x) \leq 0
\end{split}
$$

Let's define the Lagrange function:

$$
L(x, \lambda) = f(x) + \lambda g(x)
$$

The classical Karush-Kuhn-Tucker first and second-order optimality conditions for a local minimizer x^*, stated under some regularity conditions, can be written as follows.

If x^* is a local minimum of the problem described above, then there exists a unique Lagrange multiplier λ^* such that:

$$
\begin{split}
& (1) \; \nabla_x L(x^*, \lambda^*) = 0 \\
& (2) \; \lambda^* \geq 0 \\
& (3) \; \lambda^* g(x^*) = 0
\end{split}
$$

Lagrange function for inequality constraints

Combining two possible cases, we can write down the general conditions for the problem:

$$
\begin{aligned}
& f(x) \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } & g(x) \leq 0
\end{aligned}
$$

Let's define the Lagrange function:

$$
L(x, \lambda) = f(x) + \lambda g(x)
$$

The classical Karush-Kuhn-Tucker first and second-order optimality conditions for a local minimizer x^*, stated under some regularity conditions, can be written as follows.

If x^* is a local minimum of the problem described above, then there exists a unique Lagrange multiplier λ^* such that:

$$
\begin{aligned}
& (1)\ \nabla_x L(x^*, \lambda^*) = 0 \\
& (2)\ \lambda^* \geq 0 \\
& (3)\ \lambda^* g(x^*) = 0 \\
& (4)\ g(x^*) \leq 0
\end{aligned}
$$

$\lambda^* = 0,\ g(x^*) < 0$ НЕР-ВО НЕ АКТИВНО

$\lambda^* > 0,\ g(x^*) = 0$ НЕР-ВО АКТИВНО

Lagrange function for inequality constraints

Combining two possible cases, we can write down the general conditions for the problem:

$$
\begin{split}
& f(x) \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } & g(x) \leq 0
\end{split}
$$

Let's define the Lagrange function:

$$
L(x, \lambda) = f(x) + \lambda g(x)
$$

The classical Karush-Kuhn-Tucker first and second-order optimality conditions for a local minimizer x^*, stated under some regularity conditions, can be written as follows.

If x^* is a local minimum of the problem described above, then there exists a unique Lagrange multiplier λ^* such that:

$$
\begin{split}
& (1) \; \nabla_x L(x^*, \lambda^*) = 0 \\
& (2) \; \lambda^* \geq 0 \\
& (3) \; \lambda^* g(x^*) = 0 \\
& (4) \; g(x^*) \leq 0 \\
& (5) \; \forall y \in C(x^*) : \langle y, \nabla^2_{xx} L(x^*, \lambda^*) y \rangle > 0
\end{split}
$$

Lagrange function for inequality constraints

Combining two possible cases, we can write down the general conditions for the problem:

$$
f(x) \to \min_{x \in \mathbb{R}^n}
$$
$$
\text{s.t. } g(x) \leq 0
$$

Let's define the Lagrange function:

$$
L(x, \lambda) = f(x) + \lambda g(x)
$$

The classical Karush-Kuhn-Tucker first and second-order optimality conditions for a local minimizer x^*, stated under some regularity conditions, can be written as follows.

If x^* is a local minimum of the problem described above, then there exists a unique Lagrange multiplier λ^* such that:

$$
\begin{aligned}
&(1)\ \nabla_x L(x^*, \lambda^*) = 0 \\
&(2)\ \lambda^* \geq 0 \\
&(3)\ \lambda^* g(x^*) = 0 \\
&(4)\ g(x^*) \leq 0 \\
&(5)\ \forall y \in C(x^*) : \langle y, \nabla^2_{xx} L(x^*, \lambda^*) y \rangle > 0
\end{aligned}
$$

where $C(x^*) = \{y \ \in \mathbb{R}^n | \nabla f(x^*)^\top y \leq 0 \text{ and } \forall i \in I(x^*) : \nabla g_i(x^*)^T y \leq 0\}$

Lagrange function for inequality constraints

Combining two possible cases, we can write down the general conditions for the problem:

$$f(x) \to \min_{x \in \mathbb{R}^n}$$
$$\text{s.t. } g(x) \leq 0$$

Let's define the Lagrange function:

$$L(x, \lambda) = f(x) + \lambda g(x)$$

The classical Karush-Kuhn-Tucker first and second-order optimality conditions for a local minimizer x^*, stated under some regularity conditions, can be written as follows.

If x^* is a local minimum of the problem described above, then there exists a unique Lagrange multiplier λ^* such that:

$$(1)\ \nabla_x L(x^*, \lambda^*) = 0$$
$$(2)\ \lambda^* \geq 0$$
$$(3)\ \lambda^* g(x^*) = 0$$
$$(4)\ g(x^*) \leq 0$$
$$(5)\ \forall y \in C(x^*) : \langle y, \nabla^2_{xx} L(x^*, \lambda^*) y \rangle > 0$$

where $C(x^*) = \{y \ \in \mathbb{R}^n | \nabla f(x^*)^\top y \leq 0$ and $\forall i \in I(x^*) : \nabla g_i(x^*)^T y \leq 0\}$

$$I(x^*) = \{i \mid g_i(x^*) = 0\}$$

General formulation

$h_i(x) = 0 \iff \begin{cases} h_i(x) \leq 0 \\ -h_i(x) \leq 0 \end{cases}$

$$
\begin{aligned}
& f_0(x) \to \min_{x \in \mathbb{R}^n} \\
\text{s.t. } & f_i(x) \leq 0, \; i = 1, \ldots, m \\
& h_i(x) = 0, \; i = 1, \ldots, p
\end{aligned}
$$

$\min f_0(x)$

$f_i(x) \leq 0, \; i = \overline{1, m}$

This formulation is a general problem of mathematical programming.

The solution involves constructing a Lagrange function:

функция Лагранжа

$$
L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)
$$

$\lambda_i \geq 0$

множители Лагранжа

Necessary conditions

Let x^*, (λ^*, ν^*) be a solution to a mathematical programming problem with zero duality gap (the optimal value for the primal problem p^* is equal to the optimal value for the dual problem d^*). Let also the functions f_0, f_i, h_i be differentiable.

- $\nabla_x L(x^*, \lambda^*, \nu^*) = 0$

Necessary conditions

Let x^*, (λ^*, ν^*) be a solution to a mathematical programming problem with zero duality gap (the optimal value for the primal problem p^* is equal to the optimal value for the dual problem d^*). Let also the functions f_0, f_i, h_i be differentiable.

- $\nabla_x L(x^*, \lambda^*, \nu^*) = 0$
- $\nabla_\nu L(x^*, \lambda^*, \nu^*) = 0$

Necessary conditions

Let x^*, (λ^*, ν^*) be a solution to a mathematical programming problem with zero duality gap (the optimal value for the primal problem p^* is equal to the optimal value for the dual problem d^*). Let also the functions f_0, f_i, h_i be differentiable.

- $\nabla_x L(x^*, \lambda^*, \nu^*) = 0$
- $\nabla_\nu L(x^*, \lambda^*, \nu^*) = 0$
- $\lambda_i^* \geq 0, i = 1, \ldots, m$

Necessary conditions

Let x^*, (λ^*, ν^*) be a solution to a mathematical programming problem with zero duality gap (the optimal value for the primal problem p^* is equal to the optimal value for the dual problem d^*). Let also the functions f_0, f_i, h_i be differentiable.

- $\nabla_x L(x^*, \lambda^*, \nu^*) = 0$
- $\nabla_\nu L(x^*, \lambda^*, \nu^*) = 0$
- $\lambda^*_i \geq 0, i = 1, \ldots, m$
- $\lambda^*_i f_i(x^*) = 0, i = 1, \ldots, m$

Necessary conditions

$\min f_0(x)$

$f_i(x) \leq 0, \; i = 1, \ldots$

$h_j(x) = 0$

Let x^*, (λ^*, ν^*) be a solution to a mathematical programming problem with zero duality gap (the optimal value for the primal problem p^* is equal to the optimal value for the dual problem d^*). Let also the functions f_0, f_i, h_i be differentiable.

- $\nabla_x L(x^*, \lambda^*, \nu^*) = 0$
- $\nabla_\nu L(x^*, \lambda^*, \nu^*) = 0$
- $\lambda_i^* \geq 0, i = 1, \ldots, m$
- $\lambda_i^* f_i(x^*) = 0, i = 1, \ldots, m$
- $f_i(x^*) \leq 0, i = 1, \ldots, m$

$\lambda_i = 0 \quad f_i < 0$

$\lambda_i > 0 \quad f_i = 0$

Some regularity conditions

These conditions are needed to make KKT solutions the necessary conditions. Some of them even turn necessary conditions into sufficient (for example, Slater's). Moreover, if you have regularity, you can write down necessary second order conditions $\langle y, \nabla^2_{xx} L(x^*, \lambda^*, \nu^*) y \rangle \geq 0$ with *semi-definite* hessian of Lagrangian.

- **Slater's condition.** If for a convex problem (i.e., assuming minimization, f_0, f_i are convex and h_i are affine), there exists a point x such that $h(x) = 0$ and $f_i(x) < 0$ (existence of a strictly feasible point), then we have a zero duality gap and KKT conditions become necessary and sufficient.

Some regularity conditions

These conditions are needed to make KKT solutions the necessary conditions. Some of them even turn necessary conditions into sufficient (for example, Slater's). Moreover, if you have regularity, you can write down necessary second order conditions $\langle y, \nabla^2_{xx} L(x^*, \lambda^*, \nu^*) y \rangle \geq 0$ with *semi-definite* hessian of Lagrangian.

- **Slater's condition.** If for a convex problem (i.e., assuming minimization, f_0, f_i are convex and h_i are affine), there exists a point x such that $h(x) = 0$ and $f_i(x) < 0$ (existence of a strictly feasible point), then we have a zero duality gap and KKT conditions become necessary and sufficient.
- **Linearity constraint qualification.** If f_i and h_i are affine functions, then no other condition is needed.

Some regularity conditions

These conditions are needed to make KKT solutions the necessary conditions. Some of them even turn necessary conditions into sufficient (for example, Slater's). Moreover, if you have regularity, you can write down necessary second order conditions $\langle y, \nabla^2_{xx} L(x^*, \lambda^*, \nu^*) y \rangle \geq 0$ with *semi-definite* hessian of Lagrangian.

- **Slater's condition.** If for a convex problem (i.e., assuming minimization, f_0, f_i are convex and h_i are affine), there exists a point x such that $h(x) = 0$ and $f_i(x) < 0$ (existence of a strictly feasible point), then we have a zero duality gap and KKT conditions become necessary and sufficient.
- **Linearity constraint qualification.** If f_i and h_i are affine functions, then no other condition is needed.
- **Linear independence constraint qualification.** The gradients of the active inequality constraints and the gradients of the equality constraints are linearly independent at x^*.

Some regularity conditions

These conditions are needed to make KKT solutions the necessary conditions. Some of them even turn necessary conditions into sufficient (for example, Slater's). Moreover, if you have regularity, you can write down necessary second order conditions $\langle y, \nabla^2_{xx} L(x^*, \lambda^*, \nu^*) y \rangle \geq 0$ with *semi-definite* hessian of Lagrangian.

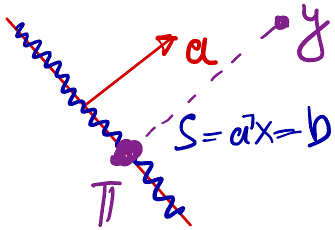
- **Slater's condition.** If for a convex problem (i.e., assuming minimization, f_0, f_i are convex and h_i are affine), there exists a point x such that $h(x) = 0$ and $f_i(x) < 0$ (existence of a strictly feasible point), then we have a zero duality gap and KKT conditions become necessary and sufficient.
- **Linearity constraint qualification.** If f_i and h_i are affine functions, then no other condition is needed.
- **Linear independence constraint qualification.** The gradients of the active inequality constraints and the gradients of the equality constraints are linearly independent at x^*.
- For other examples, see wiki.

Example. Projection onto a hyperplane

$$
\min \frac{1}{2}\|\mathbf{x} - \mathbf{y}\|^2, \quad \text{s.t.} \quad \mathbf{a}^T\mathbf{x} = b.
$$

Example. Projection onto a hyperplane

$$\min \frac{1}{2}\|\mathbf{x}-\mathbf{y}\|^2, \quad \text{s.t.} \quad \mathbf{a}^T\mathbf{x} = b.$$



Solution

Lagrangian:

1) $L = \frac{1}{2}\langle x-y, x-y\rangle + \nu(a^T x - b)$

2) $\nabla_x L = \frac{1}{2}\cdot 2(x-y) + \nu \cdot a = 0 \Rightarrow \begin{cases} x - y + \nu a = 0 \\ a^T x = b \end{cases}$ $\quad x = y - \nu a$

$\nabla_\nu L = 0 \Rightarrow a^T x = b$

$a^T(y - \nu a) = b$

$a^T y - \nu a^T a = b$

$\Rightarrow \nu = \dfrac{b - a^T y}{a^T a}$

$$x = y - \frac{b - a^T y}{a^T a} a$$

Example. Projection onto a hyperplane

$$
\min \frac{1}{2}\|\mathbf{x} - \mathbf{y}\|^2, \quad \text{s.t.} \quad \mathbf{a}^T\mathbf{x} = b.
$$

Solution

Lagrangian:

$$
L(\mathbf{x}, \nu) = \frac{1}{2}\|\mathbf{x} - \mathbf{y}\|^2 + \nu(\mathbf{a}^T\mathbf{x} - b)
$$

Example. Projection onto a hyperplane

$$
\min \frac{1}{2}\|\mathbf{x} - \mathbf{y}\|^2, \quad \text{s.t.} \quad \mathbf{a}^T\mathbf{x} = b.
$$

Solution

Lagrangian:

$$
L(\mathbf{x}, \nu) = \frac{1}{2}\|\mathbf{x} - \mathbf{y}\|^2 + \nu(\mathbf{a}^T\mathbf{x} - b)
$$

Derivative of L with respect to $\mathbf{x}$:

$$
\frac{\partial L}{\partial \mathbf{x}} = \mathbf{x} - \mathbf{y} + \nu\mathbf{a} = 0, \qquad \mathbf{x} = \mathbf{y} - \nu\mathbf{a}
$$

Example. Projection onto a hyperplane

$$\min \frac{1}{2}\|\mathbf{x} - \mathbf{y}\|^2, \quad \text{s.t.} \quad \mathbf{a}^T\mathbf{x} = b.$$

Solution

Lagrangian:

$$L(\mathbf{x}, \nu) = \frac{1}{2}\|\mathbf{x} - \mathbf{y}\|^2 + \nu(\mathbf{a}^T\mathbf{x} - b)$$

Derivative of L with respect to $\mathbf{x}$:

$$\frac{\partial L}{\partial \mathbf{x}} = \mathbf{x} - \mathbf{y} + \nu\mathbf{a} = 0, \qquad \mathbf{x} = \mathbf{y} - \nu\mathbf{a}$$

$$\mathbf{a}^T\mathbf{x} = \mathbf{a}^T\mathbf{y} - \nu\mathbf{a}^T\mathbf{a} \qquad \nu = \frac{\mathbf{a}^T\mathbf{y} - b}{\|\mathbf{a}\|^2}$$

Example. Projection onto a hyperplane

$$\min \frac{1}{2}\|\mathbf{x} - \mathbf{y}\|^2, \quad \text{s.t.} \quad \mathbf{a}^T\mathbf{x} = b.$$

Solution

Lagrangian:

$$L(\mathbf{x}, \nu) = \frac{1}{2}\|\mathbf{x} - \mathbf{y}\|^2 + \nu(\mathbf{a}^T\mathbf{x} - b)$$

Derivative of L with respect to $\mathbf{x}$:

$$\frac{\partial L}{\partial \mathbf{x}} = \mathbf{x} - \mathbf{y} + \nu\mathbf{a} = 0, \qquad \mathbf{x} = \mathbf{y} - \nu\mathbf{a}$$

$$\mathbf{a}^T\mathbf{x} = \mathbf{a}^T\mathbf{y} - \nu\mathbf{a}^T\mathbf{a} \qquad \nu = \frac{\mathbf{a}^T\mathbf{y} - b}{\|\mathbf{a}\|^2}$$

$$\mathbf{x} = \mathbf{y} - \frac{\mathbf{a}^T\mathbf{y} - b}{\|\mathbf{a}\|^2}\mathbf{a}$$

Example. Projection onto simplex

$$\min \frac{1}{2}\|x - y\|_2^2, \quad \text{s.t.} \quad x^\top 1 = 1, \quad x \geq 0. \quad x$$

KKT не позволяют найти анал. решение этой задачи

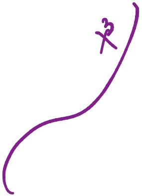
$\min \text{crossentropy}(x, y)$

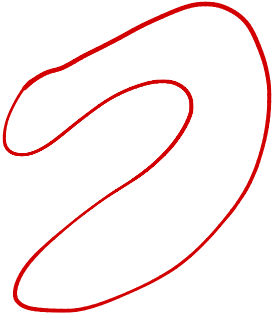
$1^\top x = 1$

$x \geq 0$

$$x_i = \frac{e^{y_i}}{\sum_{j=1}^{n} e^{y_j}}$$

$\text{softmax}(y)$





Example. Projection onto simplex

$$
\min \frac{1}{2}\|x-y\|^2, \quad \text{s.t.} \quad x^\top 1 = 1, \quad x \geq 0. \quad x
$$

KKT Conditions

The Lagrangian is given by:

$$
L = \frac{1}{2}\|x-y\|^2 - \sum_i \lambda_i x_i + \nu(x^\top 1 - 1)
$$

Example. Projection onto simplex

$$\min_x \frac{1}{2}\|x-y\|^2, \quad \text{s.t.} \quad x^\top 1 = 1, \quad x \geq 0.$$

KKT Conditions

The Lagrangian is given by:

$$L = \frac{1}{2}\|x-y\|^2 - \sum_i \lambda_i x_i + \nu(x^\top 1 - 1)$$

Taking the derivative of L with respect to x_i and writing KKT yields:

- $\frac{\partial L}{\partial x_i} = x_i - y_i - \lambda_i + \nu = 0$

Example. Projection onto simplex

$$\min_x \frac{1}{2}\|x-y\|^2, \quad \text{s.t.} \quad x^\top 1 = 1, \quad x \geq 0.$$

KKT Conditions

The Lagrangian is given by:

$$L = \frac{1}{2}\|x-y\|^2 - \sum_i \lambda_i x_i + \nu(x^\top 1 - 1)$$

Taking the derivative of L with respect to x_i and writing KKT yields:

- $\frac{\partial L}{\partial x_i} = x_i - y_i - \lambda_i + \nu = 0$
- $\lambda_i x_i = 0$

Example. Projection onto simplex

$$\min_x \frac{1}{2}\|x-y\|^2, \quad \text{s.t.} \quad x^\top 1 = 1, \quad x \geq 0.$$

KKT Conditions

The Lagrangian is given by:

$$L = \frac{1}{2}\|x-y\|^2 - \sum_i \lambda_i x_i + \nu(x^\top 1 - 1)$$

Taking the derivative of L with respect to x_i and writing KKT yields:

- $\frac{\partial L}{\partial x_i} = x_i - y_i - \lambda_i + \nu = 0$
- $\lambda_i x_i = 0$
- $\lambda_i \geq 0$

Example. Projection onto simplex

$$\min \frac{1}{2}\|x-y\|^2, \quad \text{s.t.} \quad x^\top 1 = 1, \quad x \geq 0. \quad x$$

KKT Conditions

The Lagrangian is given by:

$$L = \frac{1}{2}\|x-y\|^2 - \sum_i \lambda_i x_i + \nu(x^\top 1 - 1)$$

Taking the derivative of L with respect to x_i and writing KKT yields:

- $\frac{\partial L}{\partial x_i} = x_i - y_i - \lambda_i + \nu = 0$
- $\lambda_i x_i = 0$
- $\lambda_i \geq 0$
- $x^\top 1 = 1, \quad x \geq 0$

Example. Projection onto simplex

$$\min \frac{1}{2}\|x-y\|^2, \quad \text{s.t.} \quad x^\top 1 = 1, \quad x \geq 0. \quad x$$

KKT Conditions

The Lagrangian is given by:

$$L = \frac{1}{2}\|x-y\|^2 - \sum_i \lambda_i x_i + \nu(x^\top 1 - 1)$$

Taking the derivative of L with respect to x_i and writing KKT yields:

- $\frac{\partial L}{\partial x_i} = x_i - y_i - \lambda_i + \nu = 0$
- $\lambda_i x_i = 0$
- $\lambda_i \geq 0$
- $x^\top 1 = 1, \quad x \geq 0$

Example. Projection onto simplex

$$\min_x \frac{1}{2}\|x-y\|^2, \quad \text{s.t.} \quad x^\top 1 = 1, \quad x \geq 0.$$

KKT Conditions

The Lagrangian is given by:

$$L = \frac{1}{2}\|x-y\|^2 - \sum_i \lambda_i x_i + \nu(x^\top 1 - 1)$$

Taking the derivative of L with respect to x_i and writing KKT yields:

- $\frac{\partial L}{\partial x_i} = x_i - y_i - \lambda_i + \nu = 0$
- $\lambda_i x_i = 0$
- $\lambda_i \geq 0$
- $x^\top 1 = 1, \quad x \geq 0$

Question

Solve the above conditions in $O(n \log n)$ time.

Example. Projection onto simplex

$$\min_x \frac{1}{2}\|x - y\|^2, \quad \text{s.t.} \quad x^\top 1 = 1, \quad x \geq 0.$$

KKT Conditions

The Lagrangian is given by:

$$L = \frac{1}{2}\|x - y\|^2 - \sum_i \lambda_i x_i + \nu(x^\top 1 - 1)$$

Taking the derivative of L with respect to x_i and writing KKT yields:

- $\frac{\partial L}{\partial x_i} = x_i - y_i - \lambda_i + \nu = 0$
- $\lambda_i x_i = 0$
- $\lambda_i \geq 0$
- $x^\top 1 = 1, \quad x \geq 0$

Question

Solve the above conditions in $O(n \log n)$ time.

Question

Solve the above conditions in $O(n)$ time.

References

- Lecture on KKT conditions (very intuitive explanation) in the course “Elements of Statistical Learning” @ KTH.

References

- Lecture on KKT conditions (very intuitive explanation) in the course “Elements of Statistical Learning” @ KTH.
- One-line proof of KKT

References

- Lecture on KKT conditions (very intuitive explanation) in the course "Elements of Statistical Learning" @ KTH.
- One-line proof of KKT
- On the Second Order Optimality Conditions for Optimization Problems with Inequality Constraints

References

- Lecture on KKT conditions (very intuitive explanation) in the course "Elements of Statistical Learning" @ KTH.
- One-line proof of KKT
- On the Second Order Optimality Conditions for Optimization Problems with Inequality Constraints
- On Second Order Optimality Conditions in Nonlinear Optimization

References

- Lecture on KKT conditions (very intuitive explanation) in the course “Elements of Statistical Learning” @ KTH.
- One-line proof of KKT
- On the Second Order Optimality Conditions for Optimization Problems with Inequality Constraints
- On Second Order Optimality Conditions in Nonlinear Optimization
- Numerical Optimization by Jorge Nocedal and Stephen J. Wright.